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A Novel Class of Quadriphase Zero-Correlation Zone Sequence Sets

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SUMMARY The present paper introduces the construction of quadriphase sequences having a zero-correlation zone. For a zero-correlation zone sequence set of N sequences, each of length ℓ, the cross-correlation function and the side lobe of the autocorrelation function of the proposed sequence set are zero for the phase shifts τ within the zero-correlation zone z, such that $|\tau| \leq z$ ($\tau \neq 0$ for the autocorrelation function). The ratio $\frac{N(z+1)}{\ell}$ is theoretically limited to one. When $\ell = N(z+1)$, the sequence set is called an optimal zero-correlation sequence set. The proposed zero-correlation zone sequence set can be generated from an arbitrary Hadamard matrix of order n. The length of the proposed sequence set can be extended by sequence interleaving, where m times interleaving can generate $4n$ sequences, each of length $2^{m+3}n$. The proposed sequence set is optimal for $m = 0, 1$ and almost optimal for $m > 1$.

key words: *optimal zero-correlation zone, quadriphase, QS-CDMA, AS-CDMA*

1. Introduction

An application system for communication needs both channel (user) separation and synchronization. A sequence set having special correlation function properties can be used for the channel separation.

A sequence set having the property whereby the out-of-phase autocorrelation and cross-correlation functions are all equal to zero in a specified phase shift zone is called a *zero-correlation zone* (ZCZ) sequence set [4]. In a ZCZ sequence, the theoretical upper bound of sequence length ℓ, member size N and ZCZ width z, in which the the absolute value of the phase shift is less than or equal to z, is $N(z+1) \leq \ell$ [20]. A ZCZ sequence set that satisfies the theoretical bound of the sequence member size and the sequence period is called an optimal ZCZ sequence set [2], [5]–[11], [13], [17], [18], [21]–[23].

In the present paper, construction of a new quadriphase ZCZ sequence set is presented. The proposed sequence set has the following advantages:

1. The proposed sequence set can be constructed from an arbitrary Hadamard matrix of order n.
2. The length of the proposed sequence set can be extended by sequence interleaving, where m times interleaving can generate $4n$ sequences, each of length $2^{m+3}n$.
3. The width z of the ZCZ of the proposed sequence set is exactly equal to the theoretical bound for the ZCZ sequence set when $m \leq 1$. The width z for the autocorrelation function of the proposed sequence set is equal to the theoretical bound $2^{m+1} - 1$ for all m; the width z for the cross-correlation function of the proposed sequence set is exactly equal to the theoretical bound $2^{m+1} - 1$ for $m \leq 1$, and $2^{m-1} \cdot 3$, which is equal to $(2^{m-1} \cdot 3)/(2m + 1 - 1)$ times the theoretical bound, for $m > 1$.
4. Application systems using the proposed sequence set can be easily realized by simple hardware which can generate, transmit, and receive quadrature phase-shift keying (QPSK) signals.

After an examination of preliminary considerations in Sect. 2, a scheme for constructing the proposed sequence set is presented in Sect. 3. The properties of the proposed sequence sets are described in Sect. 4. Finally, we present concluding remarks.

2. Preliminary Considerations

A complex-number sequence of period ℓ is denoted by $\boldsymbol{v}_r = [v_{r,0}, \ldots, v_{r,\ell-1}] = [v_{r,j}]_{j=0}^{\ell-1}$. A set of N sequences $\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{N-1}\}$ is denoted by $\{\boldsymbol{v}_r | r = 0, \ldots, N-1\}$.

The ceiling of x, $\lceil x \rceil$, is the smallest integer that is not less than x, and the floor of x, $\lfloor x \rfloor$, is the largest integer that is not more than x. The quotient and modulo operations for integers a and b are denoted by $a \oslash b$ and $a \,\%\, b$, respectively, and are defined as follows:

$$a \oslash b = \begin{cases} \left\lfloor \frac{a}{b} \right\rfloor & \text{if } b > 0, \\ \left\lceil \frac{a}{b} \right\rceil & \text{if } b < 0, \end{cases} \tag{1a}$$

$$\begin{aligned} a \,\%\, b &= a - b(a \oslash b) \\ &= \begin{cases} a - b\left\lfloor \frac{a}{b} \right\rfloor & \text{if } b > 0, \\ a - b\left\lceil \frac{a}{b} \right\rceil & \text{if } b < 0. \end{cases} \end{aligned} \tag{1b}$$

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For a pair of sequences $\boldsymbol{v}_r$ and $\boldsymbol{v}_s$ of length ℓ, the periodic correlation function $\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau)$ and the aperiodic correlation function $\overset{\mathrm{A}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau)$ are respectively defined as follows:

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) = \sum_{j=0}^{\ell} v_{r,j}\ \overline{v}_{s,j+\tau}, \tag{2a}$$

$$\overset{\mathrm{A}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) = \sum_{j=0}^{\ell-\tau} v_{r,j}\ \overline{v}_{s,j+\tau}, \tag{2b}$$

where $\overline{\boldsymbol{v}}$ is the complex conjugate of $\boldsymbol{v}$.

2.1 Zero-Correlation Zone Sequence Set

If a set of sequences $\{\boldsymbol{v}_r | r = 0, \ldots, N-1\}$ of length ℓ satisfies the following conditions, then the sequence set has a ZCZ for a periodic correlation function and is denoted by $Z(\ell,\ N,\ z)$, where z is the width of the ZCZ.

For $0 < |\tau| \leq z$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_r}(\tau) = 0, \tag{3a}$$

for $r \neq s,\ \ 0 \leq |\tau| \leq z$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) = 0. \tag{3b}$$

A ZCZ sequence set that satisfies the following theoretical limit is called an optimal ZCZ sequence set. In the case of binary sequence sets, the following is true: $z \leq \frac{N}{2\ell}$ [20]. Therefore, for a binary ZCZ sequence set, $\frac{N(z+1)}{\ell}$ can be reached in the case of $z = 1$ only. Therefore, an optimal QPSK ZCZ sequence set can attain a higher ρ than can a binary ZCZ sequence set for $z > 1$.

2.2 Sequence Pair Interleaving

Here we define a sequence pair interleaving of sequence pairs $\boldsymbol{v}_r$ and $\boldsymbol{v}_r$, each of length ℓ. The sequence pair interleaving constructs a pair of sequences, $\boldsymbol{v}_r \overset{+}{\oplus} \boldsymbol{v}_s$ and $\boldsymbol{v}_r \overset{-}{\oplus} \boldsymbol{v}_s$, each of length 2ℓ, as follows:

$$\boldsymbol{v}_r \overset{+}{\oplus} \boldsymbol{v}_r = [v_{r,0},\quad v_{s,0}, \ldots, v_{r,\ell-1},\quad v_{s,\ell-1}], \tag{4a}$$

$$\boldsymbol{v}_r \overset{-}{\oplus} \boldsymbol{v}_r = [v_{r,0}, -v_{s,0}, \ldots, v_{r,\ell-1}, -v_{s,\ell-1}]. \tag{4b}$$

For an even number n, we can construct a different ZCZ sequence set of n sequences of length 2ℓ from a $PZCZ\ell nz$ set of n sequences of length 2ℓ by sequence pair interleaving.

Here we show the facts of the correlation function of $\boldsymbol{v}_r \overset{+}{\oplus} \boldsymbol{v}_s$ and $\boldsymbol{v}_r \overset{-}{\oplus} \boldsymbol{v}_s$. For simplicity, we denote $\boldsymbol{v}_r \overset{+}{\oplus} \boldsymbol{v}_s$ and $\boldsymbol{v}_r \overset{-}{\oplus} \boldsymbol{v}_s$ by $\boldsymbol{w}^{(+)}$ and $\boldsymbol{w}^{(-)}$, respectively.

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(+)},\,\boldsymbol{w}^{(+)}}(2\tau) \\
&= \sum_{i=0}^{\ell-1} w^{(+)}_{2i}\overline{w^{(+)}}_{2i+2\tau} + \sum_{i=0}^{\ell-1} w^{(+)}_{2i+1}\overline{w^{(+)}}_{2i+1+2\tau} \\
&= \sum_{i=0}^{\ell-1} v_{r,i}\overline{v}_{r,i+\tau} + \sum_{i=0}^{\ell-1} v_{s,i}\overline{v}_{s,i+\tau} \\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_r}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_s}(\tau).
\end{aligned} \tag{5a}$$

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(+)},\,\boldsymbol{w}^{(+)}}(2\tau+1) \\
&= \sum_{i=0}^{\ell-1} w^{(+)}_{2i}\overline{w^{(+)}}_{2i+2\tau+1} + \sum_{i=0}^{\ell-1} w^{(+)}_{2i+1}\overline{w^{(+)}}_{2i+1+2\tau+1} \\
&= \sum_{i=0}^{\ell-1} v_{r,i}\overline{v}_{s,i+\tau} + \sum_{i=0}^{\ell-1} v_{s,i}\overline{v}_{r,i+1+\tau} \\
&= \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_r}(\tau+1).
\end{aligned} \tag{5b}$$

Similarly, we can obtain the following:

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(+)},\,\boldsymbol{w}^{(-)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_r}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_s}(\tau), \tag{5c}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(+)},\,\boldsymbol{w}^{(-)}}(2\tau+1) = -\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_r}(\tau+1), \tag{5d}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(-)},\,\boldsymbol{w}^{(+)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_r}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_s}(\tau), \tag{5e}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(-)},\,\boldsymbol{w}^{(+)}}(2\tau+1) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_r}(\tau+1), \tag{5f}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(-)},\,\boldsymbol{w}^{(-)}}(2\tau) = \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_r}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_s}(\tau), \tag{5g}$$

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{w}^{(-)},\,\boldsymbol{w}^{(-)}}(2\tau+1) = -\overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_r,\,\boldsymbol{v}_s}(\tau) - \overset{\mathrm{P}}{\theta}_{\boldsymbol{v}_s,\,\boldsymbol{v}_r}(\tau+1). \tag{5h}$$

3. Sequence Construction

The proposed scheme for sequence construction is presented in this section.

A set of complex-number sequences having a ZCZ can be constructed from an Hadamard matrix $\boldsymbol{H}$ of order n. The i-th row of the Hadamard matrix $\boldsymbol{H}$ is denoted by $\boldsymbol{h}_i = [h_{i,0}, \ldots, h_{i,n-1}]$.

First, a set of $4n$ sequences $\boldsymbol{g}_{4r+s}$, each of length $4n$, is constructed from the Hadamard matrix $\boldsymbol{H}$ of order n.

Next, a set of $4n$ sequences $\{\boldsymbol{f}^{(0)}_*\}$, each of length $4n$, is constructed from the sequence set $\{\boldsymbol{g}_*\}$.

Finally, sets of $4n$ sequences $\{\boldsymbol{f}^{(m+1)}_*\}$, each of length $2^{(m+1)+3}$, are constructed by the interleaving of $\{\boldsymbol{f}^{(m)}_*\}$, each of length 2^{m+3}, recursively.

3.1 Construction of a Sequence Set $\{\boldsymbol{g}_*\}$

From the Hadamard matrix $\boldsymbol{H}$ of order n, a set of $4n$ sequences $\boldsymbol{g}_{4r+s}$, each of length $4n$, $= [g_{4r+s,\,0}, \ldots g_{4r+s,\,4n-1}]$ is constructed as follows:

For $0 \leq r < n,\ 0 \leq s < 4$,

$$\boldsymbol{g}_{4r+s} = [\boldsymbol{h}_r,\ \imath^s\boldsymbol{h}_r,\ \imath^{2s}\boldsymbol{h}_r,\ \imath^{3s}\boldsymbol{h}_r], \tag{6a}$$

where $\imath = \sqrt{-1}$. Equation (6a) can be formulated as follows:

For $0 \leq r < n,\ 0 \leq s < 4,\ 0 \leq j < 4n$,

$$g_{4r+s,\,j} = \iota^{s(j \oslash n)} h_{r,\,j\%n}. \tag{6b}$$

Since j can be expressed by $j = i+nk$, $0 \le i < n$, $0 \le k < 4$, the correlation function of $\boldsymbol{g}_r$, each of length $4n$ for phase shift $0 \le \tau < n$, can be computed as follows:

For $0 \le r, r' < n,\ 0 \le s, s' < 4$,

$$\begin{aligned}
&\forall\tau, \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+s},\,\boldsymbol{g}_{4r'+s'}}(\tau)\\
&= \sum_{i=0}^{n-1}\sum_{k=0}^{3} g_{4r+s,\,i+nk}\,\overline{g}_{4r'+s',\,i+nk+\tau}\\
&= \sum_{i=0}^{n-1}\sum_{k=0}^{3} \iota^{s((i+nk)\oslash n)} h_{r,(i+nk)\%n}\\
&\qquad\qquad \iota^{-s'((i+nk+\tau)\oslash n)} h_{r',(i+nk+\tau)\%n}\\
&= \sum_{i=0}^{n-1}\sum_{k=0}^{3} \iota^{sk} h_{r,i}\, \iota^{-s'(k+(i+\tau)\oslash n)} h_{r',(i+\tau)\%n}\\
&= \sum_{i=0}^{n-\tau-1}\sum_{k=0}^{3} \iota^{sk} h_{r,i}\, \iota^{-s'(k+\tau\oslash n)} h_{r',(i+\tau)\%n}\\
&+ \sum_{i=n-\tau}^{n-1}\sum_{k=0}^{3} \iota^{sk} h_{r,i}\, \iota^{-s'(k+1+\tau\oslash n)} h_{r',(i+\tau)\%n}\\
&= \sum_{k=0}^{3} \iota^{(s-s')k-s'(\tau\oslash n)} \overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(\tau)\\
&+ \sum_{k=0}^{3} \iota^{(s-s')k-s'(1+(\tau\oslash n))} \overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(n-\tau).
\end{aligned} \tag{7}$$

From Eq. (7), the correlation function of $4n$ sequences $\boldsymbol{g}_r$, each of length $4n$, satisfies the following:

For $0 \le r, r' < n,\ 0 \le s, s' < 4$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+s},\,\boldsymbol{g}_{4r+s}}(0) = \begin{cases} 4n & \text{if } (r,s) = (r',s'), \\ 0 & \text{if } (r,s) \ne (r',s'). \end{cases} \tag{8a}$$

For $0 \le r, r' < n,\ 0 \le s \ne s' < 4,\ (r,s) \ne (r',s')$, $\forall\tau$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+s},\,\boldsymbol{g}_{4r+s'}}(\tau) = 0. \tag{8b}$$

Similarly, we have the following:

For $0 \le r, r' < n,\ 0 \le s \ne s' < 4,\ 0 \le \tau < n$,

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+s},\,\boldsymbol{g}_{4r'+s'}}(\tau)\\
&\quad = 4\left(\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(\tau) + \iota^{s'}\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(n-\tau)\right).
\end{aligned} \tag{9a}$$

From Eq. (9a), we can obtain the following:

For $0 \le r, r' < n,\ 0 \le \tau < n$,

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r},\,\boldsymbol{g}_{4r'}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+2},\,\boldsymbol{g}_{4r'+2}}(\tau) =\\
&\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+1},\,\boldsymbol{g}_{4r'+1}}(\tau) + \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{4r+3},\,\boldsymbol{g}_{4r'+3}}(\tau) =\\
&\quad 8\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(\tau).
\end{aligned} \tag{9b}$$

Hereafter, we decompose s as $2p+q$, $0 \le p, q < 2$ ($s = 2p+q$). For a fixed number n, we can recursively construct a series of sets $\{\boldsymbol{f}^{(m)}_{4r+2p+q} | 0 \le r < n, 0 \le p, q < 2\}$ of $4n$ sequences for $m \ge 0$, as shown in the following subsection.

3.2 Construction of a Sequence Set $\{\boldsymbol{f}^{(0)}_*\}$

A sequence set $\{\boldsymbol{f}^{(0)}_{4r+2p+q}\}$ is constructed from the sequence set $\{\boldsymbol{g}_{4r+2p+q} | 0 \le r < n, 0 \le p, q < 2\}$. The sequences $\boldsymbol{f}^{(0)}_{(4r+0)+q}$ and $\boldsymbol{f}^{(0)}_{(4r+2)+q}$ are constructed by the interleaving of sequence pairs $\boldsymbol{g}_{(4r+0)+q}$ and $\boldsymbol{g}_{(4r+2)+q}$ as follows:

For $0 \le r < n,\ 0 \le s = 2p+q < 4,\ 0 \le p, q < 2$

$$\begin{aligned}
\boldsymbol{f}^{(0)}_{4r+s} &= \boldsymbol{f}^{(0)}_{4r+2p+q}\\
&= \begin{cases} \boldsymbol{g}_{(4r+0)+q} \overset{+}{\oplus} \boldsymbol{g}_{(4r+2)+q} & \text{if } p = 0, \\ \boldsymbol{g}_{(4r+0)+q} \overset{-}{\oplus} \boldsymbol{g}_{(4r+2)+q} & \text{if } p = 1. \end{cases}
\end{aligned} \tag{10}$$

The sequence $\boldsymbol{f}^{(0)}_{4r+2p+q}$ is $2 \cdot 4n = 8n$ in length, and the member size of the sequence set $\{\boldsymbol{f}^{(0)}_{4r+2p+q}\}$ is $4n$.

3.3 Recursive Construction of a Sequence Set $\{\boldsymbol{f}^{(m+1)}_*\}$ from $\{\boldsymbol{f}^{(m)}_*\}$

We can generate a series of sequence sets $\{\boldsymbol{f}^{(m+1)}_{4r+2p+q}\}$ by the interleaving of $\{\boldsymbol{f}^{(m)}_{4r+2p+q}\}$ recursively.

For $m \ge 1$, we assume the construction of $\{\boldsymbol{f}^{(m)}_{4r+2p+q} | 0 \le r < n,\ 0 \le p, q < 2\}$, each of length $2^{m+3}n$ ($8n$ for $m = 0$). Then, $\{\boldsymbol{f}^{(m+1)}_r | r = 0, \ldots, 4n-1\}$ is generated as follows:

For $0 \le r < n,\ 0 \le p, q < 2$

$$\begin{aligned}
&\boldsymbol{f}^{(m+1)}_{4r+2p+q}\\
&\quad = \begin{cases} \boldsymbol{f}^{(m)}_{(4r+2p)+0} \overset{+}{\oplus} \boldsymbol{f}^{(m)}_{(4r+2p)+1} & \text{if } q = 0, \\ \boldsymbol{f}^{(m)}_{(4r+2p)+0} \overset{-}{\oplus} \boldsymbol{f}^{(m)}_{(4r+2p)+1}, & \text{if } q = 1. \end{cases}
\end{aligned} \tag{11}$$

Note that the proposed sequence construction uses the sequence pairs $(\boldsymbol{f}^{(0)}_{4r+0}, \boldsymbol{f}^{(0)}_{4r+2})$ and $(\boldsymbol{f}^{(0)}_{4r+1}, \boldsymbol{f}^{(0)}_{4r+3})$ for $m = 0$ in Eq. (10) and the sequence construction uses the sequence pairs $(\boldsymbol{f}^{(m)}_{4r+0}, \boldsymbol{f}^{(m)}_{4r+1})$ and $(\boldsymbol{f}^{(m)}_{4r+2}, \boldsymbol{f}^{(m)}_{4r+3})$ for $m > 0$ in Eq. (11).

The length of $\boldsymbol{f}^{(m+1)}_{4r+2p+q}$ is twice that of $\boldsymbol{f}^{(m)}_{4r+2p+q}$ and is equal to $2^{(m+1)+3}n$.

In the following section, we present the properties of the constructed sequences.

4. Properties of the Constructed Sequences

The sequence set $\{\boldsymbol{f}^{(m)}_{4r+2p+q} | 0 \le r < n, 0 \le p, q < 2\}$ has a ZCZ for the periodic correlation function, $\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}^{(m)}_{4r+2p+q},\,\boldsymbol{f}^{(m)}_{4r'+2p'+q'}}(\tau)$ for phase shift τ.

4.1 Properties of the Proposed Sequence Set for $0 \le m \le 1$

The ZCZ of the sequence set $\{\boldsymbol{f}_r^{(m)}\}$ is stretched via doubling of the interleaving sequence length.

We then have the following theorem.

Theorem 1. The periodic correlation function of $\{\boldsymbol{f}_{4r+2p+q}^{(m)}\}$, each of length $2^{m+3}n$, has a ZCZ from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$ for $0 \le m \le 1$. That is,

For $0 \le m \le 1$,

$$\forall r,\ p,\ q\ |\tau| \le 2^{m+1}-1,$$
$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(m)},\,\boldsymbol{f}_{4r+2p+q}^{(m)}}(\tau) = \begin{cases} 2^{m+3}n & \text{if } \tau = 0, \\ 0 & \text{if } 0 < |\tau| \le 2^{m+1}-1, \end{cases} \tag{12a}$$

and

$$\forall 4r+2p+q \ne 4r'+2p'+q',\ |\tau| \le 2^{m+1}-1,$$
$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(m)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(m)}}(\tau) = 0. \tag{12b}$$

Proof. Here, we compute the correlation function of the proposed sequences $\boldsymbol{f}_{4r+2p+q}^{(m)}$ to show Theorem 1 by using Eqs. (5a)–(5h), (6a), (6b), (8a), (8b), and (10). We consider $\tau \ge 0$ without any loss of generality. From Eqs. (10), (5a), (5c), (5e), and (5g), we obtain the following:

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(0)}}(0) \\ &= \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+0)+q},\,\boldsymbol{g}_{(4r'+0)+q'}}(0) \\ &+ (-1)^{(p-p')}\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+2)+q},\,\boldsymbol{g}_{(4r'+2)+q'}}(0). \end{aligned} \tag{13a}$$

From Eq. (8a), we obtain the following:

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(0)}}(0) \\ &= \begin{cases} 8n & \text{if } (r,p,q) = (r',p',q'), \\ 0 & \text{if } (r,p,q) \ne (r',p',q'). \end{cases} \end{aligned} \tag{13b}$$

Similarly, we obtain the following from Eqs. (5b), (5d), (5f), (5h), and (10):

For $0 \le r, r' < n,\ 0 \le p, p', q, q' < 2$,

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(0)}}(1) \\ &= (-1)^{p'}\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+0)+q},\,\boldsymbol{g}_{(4r'+2)+q'}}(0) \\ &+ (-1)^{p}\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+2)+q},\,\boldsymbol{g}_{(4r'+0)+q'}}(1). \end{aligned} \tag{14a}$$

From Eqs. (8a) and (8b), we obtain the following:

For $0 \le r, r' < n,\ 0 \le p, p', q, q' < 2$,

$$\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(0)}}(1) = 0. \tag{14b}$$

We also obtain the following from Eqs. (10), (5a), (5c), (5e), and (5g):

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(0)}}(2) \\ &= \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+0)+q},\,\boldsymbol{g}_{(4r'+0)+q'}}(1) \\ &+ (-1)^{(p-p')}\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+2)+q},\,\boldsymbol{g}_{(4r'+2)+q'}}(1). \end{aligned} \tag{15a}$$

From Eqs. (8b) and (15a), we have the following:

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+0}^{(0)},\,\boldsymbol{f}_{4r'+2p'+1}^{(0)}}(2) \\ &= \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+0)+0},\,\boldsymbol{g}_{(4r'+0)+1}}(1) \\ &+ (-1)^{(p-p')}\overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+2)+0},\,\boldsymbol{g}_{(4r'+2)+1}}(1) \\ &= 0. \end{aligned} \tag{15b}$$

From Eqs. (9b) and (15a), we have the following:

For $p = p'$,

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(0)},\,\boldsymbol{f}_{4r'+2p+q'}^{(0)}}(2) \\ &= \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+0)+q},\,\boldsymbol{g}_{(4r'+0)+q'}}(1) \\ &+ \overset{\mathrm{P}}{\theta}_{\boldsymbol{g}_{(4r+2)+q},\,\boldsymbol{g}_{(4r'+2)+q'}}(1) \\ &= 8\overset{\mathrm{A}}{\theta}_{\boldsymbol{h}_r,\,\boldsymbol{h}_{r'}}(\tau). \end{aligned} \tag{16a}$$

Then we obtain

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{(4r+0)+q}^{(0)},\,\boldsymbol{f}_{(4r'+0)+q'}^{(0)}}(2) = \\ &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{(4r+2)+q}^{(0)},\,\boldsymbol{f}_{(4r'+2)+q'}^{(0)}}(2). \end{aligned} \tag{16b}$$

For $m \ge 0$, we can compute the correlation functions of the proposed sequence set as follows.

From Eqs. (5a), (5c), (5e), (5g), and (11), we obtain the following:

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(m+1)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(m+1)}}(2\tau) \\ &= \overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+0}^{(m)},\,\boldsymbol{f}_{4r'+2p'+0}^{(m)}}(\tau) \\ &+ (-1)^{(q-q')}\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+1}^{(m)},\,\boldsymbol{f}_{4r'+2p'+1}^{(m)}}(\tau). \end{aligned} \tag{17a}$$

Also, from Eqs. (5b), (5d), (5f), (5h), and (11), we obtain the following:

$$\begin{aligned} &\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+q}^{(m+1)},\,\boldsymbol{f}_{4r'+2p'+q'}^{(m+1)}}(2\tau+1) \\ &= (-1)^{q'}\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+0}^{(m)},\,\boldsymbol{f}_{4r'+2p'+1}^{(m)}}(\tau) \\ &+ (-1)^{q}\overset{\mathrm{P}}{\theta}_{\boldsymbol{f}_{4r+2p+1}^{(m)},\,\boldsymbol{f}_{4r'+2p'+0}^{(m)}}(\tau+1). \end{aligned} \tag{17b}$$

Then we obtain the following from Eqs. (13b), (17a), and, (17b):

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+q},\, f^{(1)}_{4r'+2p'+q'}}(0)\\
&= \overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+0},\, f^{(0)}_{4r'+2p'+0}}(0)\\
&+ (-1)^{(q-q')}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+1},\, f^{(0)}_{4r'+2p'+1}}(0)\\
&= \begin{cases} 16n & \text{if } 4r+2p+q = 4r'+2p'+q',\\ 0 & \text{if } 4r+2p+q \neq 4r'+2p'+q'. \end{cases}
\end{aligned} \tag{18a}$$

From Eqs. (5b), (5d), (5f), (5h), and (10), we obtain

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+q},\, f^{(1)}_{4r'+2p'+q'}}(1)\\
&= (-1)^{q'}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+0},\, f^{(0)}_{4r'+2p'+1}}(0)\\
&+ (-1)^{q}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+1},\, f^{(0)}_{4r'+2p'+0}}(1)\\
&= 0.
\end{aligned} \tag{18b}$$

From Eqs. (5b), (5d), (5f), (5h), and (10), we obtain

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+q},\, f^{(1)}_{4r'+2p'+q'}}(2)\\
&= \overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+0},\, f^{(0)}_{4r'+2p'+0}}(1)\\
&+ (-1)^{(q-q')}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+1},\, f^{(0)}_{4r'+2p'+1}}(1)\\
&= 0.
\end{aligned} \tag{18c}$$

Next, we can compute the correlation function for phase shift $\tau = 3$ from Eqs. (5a)–(5h), and eqrs0def as follows:

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+q},\, f^{(1)}_{4r'+2p'+q'}}(3)\\
&= (-1)^{q'}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+0},\, f^{(0)}_{4r'+2p'+1}}(1)\\
&+ (-1)^{q}\overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+1},\, f^{(0)}_{4r'+2p'+0}}(2).
\end{aligned} \tag{18d}$$

From Eqs. (14b) and (15b),

$$\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+q},\, f^{(1)}_{4r'+2p'+q'}}(3) = 0. \tag{18e}$$

Then, from Eqs. (10) and (5c), we can compute the correlation function for $\tau = 4$ as follows:

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+0},\, f^{(1)}_{4r+2p+1}}(4)\\
&= \overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+0},\, f^{(0)}_{4r+2p+0}}(2)\\
&- \overset{\mathrm{P}}{\theta}_{f^{(0)}_{4r+2p+1},\, f^{(0)}_{4r+2p+1}}(2).
\end{aligned} \tag{18f}$$

From Eqs. (16b) and (18f),

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(1)}_{4r+2p+0},\, f^{(1)}_{4r'+2p'+1}}(4)\\
&= 0.
\end{aligned} \tag{18g}$$

Thus, from Eqs. (13b), (14b), and Eqs. (18a)–(18e), Theorem 1 is proved. □

4.2 Properties of the Proposed Sequence Set for $m \geq 1$

We also have the following theorem:

Theorem 2. For $m \geq 1$, the autocorrelation function of the generated sequences, $\{f^{(m)}_r | r = 0, \ldots, 4n-1\}$, has a ZCZ from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$, and the cross-correlation function of the sequences has a ZCZ from $-2^{m-1}3$ to $2^{m-1}3$, as in the following:

For $m \geq 1$,

$$\begin{aligned}
&\forall r, p, q\ \forall |\tau| \leq 2^{m+1}-1,\\
&\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+q},\, f^{(m)}_{4r+2p+q}}(\tau)\\
&= \begin{cases} 2^{(m+3)}n, & \text{if } \tau = 0,\\ 0, & 0 < |\tau| < 2^{m+1}-1. \end{cases}
\end{aligned} \tag{19a}$$

For $m \geq 1$,

$$\begin{aligned}
&\forall (r,p,q) \neq (r',p',q'),\ \forall \tau,\ |\tau| \leq 2^{m-1}3,\\
&\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+q},\, f^{(m)}_{4r'+2p'+q'}}(\tau) = 0.
\end{aligned} \tag{19b}$$

For $m \geq 1$,

$$\forall r, p,$$

$$\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},\, f^{(m)}_{4r+2p+1}}(2^{m+1}) = 0, \tag{19c}$$

$$\begin{aligned}
&\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},\, f^{(m)}_{4r+2p+0}}(2^{m+1}) =\\
&\quad = \overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},\, f^{(m)}_{4r+2p+1}}(2^{m+1}).
\end{aligned} \tag{19d}$$

Proof. Proof of Theorem 2 For $m \geq 1$, we prove Theorem 2 by mathematical induction on $m \geq 1$. Here, we consider $\tau \geq 0$ without any loss of generality. For $m = 1$, Theorem 2 is identical to Theorem 1. Thus, Theorem 2 is proved for $m = 1$. For $m \geq 1$, we assume that Theorem 2 is satisfied for $\{f^{(1)}_{4r+2p+q} | 0 \leq r < 4, 0 \leq p, q < 2\}, \ldots, \{f^{(m-1)}_{4r+2p+q} | 0 \leq r < 4, 0 \leq p, q < 2\}$. From Eqs. (5g), (5g), (11), and (19a), we can compute the autocorrelation function of the proposed sequences as follows:

$$\begin{aligned}
&\forall r, p, q\ \forall \tau \leq 2^{m+1}-1,\\
&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},\, f^{(m+1)}_{4r+2p+q}}(2\tau)\\
&= \overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},\, f^{(m)}_{4r+2p+0}}(\tau)\\
&+ \overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},\, f^{(m)}_{4r+2p+1}}(\tau)\\
&= \begin{cases} 2^{(m+3)+1}n, & \text{if } \tau = 0,\\ 0, & \text{if } 0 < \tau \leq 2^{m+1}-1. \end{cases}
\end{aligned} \tag{20a}$$

From Eqs. (5b), (5h), (11), and (19a), we obtain the following:

$$\begin{aligned}&\forall r,p,q,\ \forall\tau\le 2^{m-1}3,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r+2p+q}}(2\tau+1)\\&=(-1)^q\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r+2p+1}}(\tau)\\&+(-1)^q\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r+2p+0}}(\tau+1)\\&=0.\end{aligned}\tag{20b}$$

From Eqs. (5c), (5e), (11), and (19c), we can compute the cross-correlation function of the proposed sequences as follows:

$$\begin{aligned}&\forall r,p,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+0},f^{(m+1)}_{4r+2p+1}}(2^{(m+1)+1})\\&=\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r+2p+0}}(2^{m+1})\\&-\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r+2p+1}}(2^{m+1})=0.\end{aligned}\tag{21a}$$

From Eqs. (5b), (5h), (11), and (19c), we can compute the autocorrelation function of the proposed sequences as follows:

$$\begin{aligned}&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+0},f^{(m+1)}_{4r+2p+0}}(2^{(m+1)+1})\\&=\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r+2p+0}}(2^{m+1})\\&+\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r+2p+1}}(2^{m+1}).\end{aligned}\tag{22a}$$

Similarly, we obtain

$$\begin{aligned}&=\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+1},f^{(m+1)}_{4r+2p+1}}(2^{(m+1)+1})\\&=\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r+2p+0}}(2^{m+1})\\&+\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r+2p+1}}(2^{m+1}).\end{aligned}\tag{22b}$$

Then we have

$$\begin{aligned}&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+0},f^{(m+1)}_{4r+2p+0}}(2^{(m+1)+1})\\&=\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+1},f^{(m+1)}_{4r+2p+1}}(2^{(m+1)+1}).\end{aligned}\tag{22c}$$

From Eqs. (5b), (5h), (11), and (19c), we can compute the autocorrelation function of the proposed sequences as follows:

$$\begin{aligned}&\forall r,p,q,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r+2p+q}}(2^{(m+1)+1}-1)\\&=(-1)^q\Big(\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r+2p+1}}(2^{m+1})\\&+\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r+2p+1}}(2^{m+1})\Big).\end{aligned}\tag{23a}$$

From Eqs. (20b), (21a), and (22c), we consequently obtain the following:

$$\begin{aligned}&\forall r,p,q\ \forall\tau\le 2^{(m+1)+1}-1,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r+2p+q}}(\tau)\\&=\begin{cases}2^{((m+1)+3)}n, & \text{if } \tau=0,\\0, & \text{if } 0<\tau\le 2^{(m+1)+1}-1.\end{cases}\end{aligned}\tag{23b}$$

Similarly, we also obtain the following from Eqs. (11) and (5a)–Eqs. (5h):

$$\begin{aligned}&\forall(r,p,q)\ne(r',p',q'),\ 0\le\tau<2^{m-1}3,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r'+2p'+q'}}(2\tau)\\&=\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r'+2p'+0}}(\tau)\\&+(-1)^{q-q'}\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r'+2p'+0}}(\tau)\\&=0,\end{aligned}\tag{24a}$$

For $0\le r,r'<n,\ 0\le p,p',q,q'<2,$

$$\begin{aligned}&(r,p,q)\ne(r',p',q'),\ 0\le\tau<2^{m-1}3-1,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r'+2p'+q'}}(2\tau+1)\\&=(-1)^{q'}\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+0},f^{(m)}_{4r'+2p'+1}}(\tau)\\&+(-1)^{q}\overset{\mathrm{P}}{\theta}_{f^{(m)}_{4r+2p+1},f^{(m)}_{4r'+2p'+0}}(\tau+1)\\&=0.\end{aligned}\tag{24b}$$

Then we have the following:

$$\begin{aligned}&\forall(r,p,q)\ne(r',p',q'),\ \forall\tau\le 2^{(m+1)-1}3,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r'+2p'+q'}}(\tau)=0.\end{aligned}\tag{24c}$$

From Eqs. (23b), (24c), (21a), and (22c), we obtain the following:

$$\begin{aligned}&\forall r,p,q\ \forall\tau\le 2^{(m+1)+1}-1,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r+2p+q}}(\tau)\\&=\begin{cases}2^{((m+1)+3)}n, & \text{if } \tau=0,\\0, & 0<|\tau|<2^{(m+1)+1}-1.\end{cases}\end{aligned}\tag{25a}$$

$$\begin{aligned}&\forall(r,p,q)\ne(r',p',q'),\ \forall\tau,\ |\tau|\le 2^{(m+1)-1}3,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+q},f^{(m+1)}_{4r'+2p'+q'}}(\tau)=0.\end{aligned}\tag{25b}$$

$$\begin{aligned}&\forall r,p,\\&\overset{\mathrm{P}}{\theta}_{f^{(m+1)}_{4r+2p+0},f^{(m+1)}_{4r+2p+1}}(2^{(m+1)+1})=0,\end{aligned}\tag{25c}$$

$$\overset{P}{\theta}_{f^{(m+1)}_{4r+2p+0},\, f^{(m+1)}_{4r+2p+0}}(2^{(m+1)+1}) = \\ = \overset{P}{\theta}_{f^{(m+1)}_{4r+2p+1},\, f^{(m+1)}_{4r+2p+1}}(2^{(m+1)+1}). \tag{25d}$$

Thus, Theorem 2 is proved. □

□

Theorems 1 and 2 indicate that the sequence set $\{f_r^{(m)} | r = 0, \ldots, 4n-1\}$ is $Z(2^{m+3}n,\ 4n,\ 2^{m+1}-1)$ for $0 \le m \le 1$ and $Z(2^{m+3}n,\ 4n,\ 2^{m-1}3))$ for $m \ge 1$.

The theoretical upper bound of the sequence member size of a $Z(L,\ N,\ z)$ sequence set is $\frac{L}{z+1}$ [3], [12], [17], [20]. Theorem 1 indicates that the width of the ZCZ of the proposed sequence set is exactly equal to the theoretical bound for the ZCZ sequence set, from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$, for $m \le 1$. Theorem 2 indicates that the width of the ZCZ of the autocorrelation function of the proposed sequence is equal to the theoretical upper bound, from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$.

We can generate the following ZCZ sequence set $Z(32,\ 8,\ 3)$ from an Hadamard matrix of order 2:

$$\begin{aligned}
f_0^{(1)} &= [+,+,+,+,+,+,+,+,+,\imath,-,\bar{\imath},+,\imath,-,\bar{\imath},\\
&\quad +,-,+,-,+,-,+,-,+,\bar{\imath},-,\imath,+,\bar{\imath},-,\imath],\\
f_1^{(1)} &= [+,-,+,-,+,-,+,-,+,\bar{\imath},-,\imath,+,\bar{\imath},-,\imath,\\
&\quad +,+,+,+,+,+,+,+,+,\imath,-,\bar{\imath},+,\imath,-,\bar{\imath}],\\
f_2^{(1)} &= [+,+,-,-,+,+,-,-,+,\imath,+,\imath,+,\imath,+,\imath,\\
&\quad +,-,-,+,+,-,-,+,+,\bar{\imath},+,\bar{\imath},+,\bar{\imath},+,\bar{\imath}],\\
f_3^{(1)} &= [+,-,-,+,+,-,-,+,+,\bar{\imath},+,\bar{\imath},+,\bar{\imath},+,\bar{\imath},\\
&\quad +,+,-,-,+,+,-,-,+,\imath,+,\imath,+,\imath,+,\imath],\\
f_4^{(1)} &= [+,+,+,+,-,-,-,-,+,\imath,-,\bar{\imath},-,\bar{\imath},+,\imath,\\
&\quad +,-,+,-,-,+,-,+,+,\bar{\imath},-,\imath,-,\imath,+,\bar{\imath}],\\
f_5^{(1)} &= [+,-,+,-,-,+,-,+,+,\bar{\imath},-,\imath,-,\imath,+,\bar{\imath},\\
&\quad +,+,+,+,-,-,-,-,+,\imath,-,\bar{\imath},-,\bar{\imath},+,\imath],\\
f_6^{(1)} &= [+,+,-,-,-,-,+,+,+,\imath,+,\imath,-,\bar{\imath},-,\bar{\imath},\\
&\quad +,-,-,+,-,+,+,-,+,\bar{\imath},+,\bar{\imath},-,\imath,-,\imath],\\
f_7^{(1)} &= [+,-,-,+,-,+,+,-,+,\bar{\imath},+,\bar{\imath},-,\imath,-,\imath,\\
&\quad +,+,-,-,-,-,+,+,+,\imath,+,\imath,-,\bar{\imath},-,\bar{\imath}],
\end{aligned}$$

where $\imath = \sqrt{-1}$ and $\bar{\imath} = -\imath$.

4.3 Communications Applications

In a DS-CDMA system, a sequence set is used for channel separation. However, inaccurate synchronization and multi-path propagation cause a time delay that destroys the orthogonality of the channel separation. In an approximately synchronized CDMA (AS-CDMA) system or quasi CDMA (QS-CDMA) system, a ZCZ sequence set allows co-channel interference to be eliminated [16]. The application of a ternary ZCZ sequence set to an AS-CDMA system was demonstrated in [17].

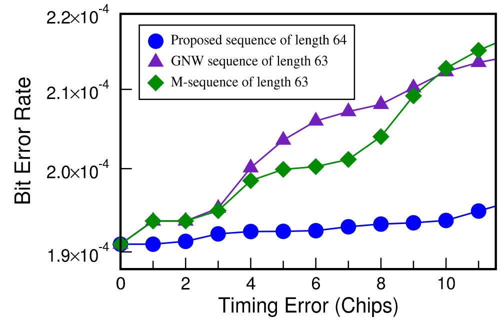

Fig. 1 Bit error eate with respect to the timing error.

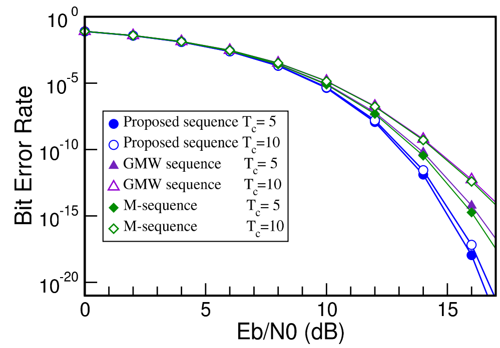

Fig. 2 Bit error rate in terms of E_b/N_0.

The proposed ternary ZCZ sequence set can be applied to AS-CDMA in the same manner.

To show the performance of the proposed sequence sets for an AS-CDMA system, the bit error rate (BER) of an AS-CDMA system using the proposed sequence set ($Z(64,\ 4,\ 12)$) ($n = 1$ and $m = 3$) is estimated as shown in [1], [14], [15]

Figure 1 shows the BER performance with respect to the timing error for the case of $E_b/N_0 = 8$ dB. The performance of the proposed sequence set, which is $Z(84,\ 6,\ 13)$, is compared with that of a GMW sequence of length 63 and an M-sequence of length 63. The BER performance in terms of E_b/N_0 is shown in Fig. 2. These figures demonstrate the advantage of the proposed sequence set when applied to an AS-CDMA system with timing error. Tang *et al.* proposed a sequence set having a low correlation sequence zone [19].

5. Conclusions

A new construction scheme of a quadriphase ZCZ sequence set was presented. The proposed sequence set $\{f_r^{(m)} | r = 0, \ldots, 4n-1\}$ is constructed from an Hadamard matrix of order n for a non-negative integer $m \ge 0$.

The proposed sequence construction can generate sequence sets $\{f_r^{(m)} | r = 0, \ldots, 2n-1\}$ having $Z(2^{m+4}n,\ 8n,\ 2^{m+1}-1))$ for $0 \le m \le 1$ and $Z(2^{m+4}n,\ 8n,\ 2^{m-1}3))$, for $m \ge 1$, for given Hadamard matrix of order n. The width of the ZCZ of the proposed sequence set is exactly equal to the theoretical bound for the ZCZ sequence set, from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$ for $m \le 1$. A quadriphase sequence can be easily applied to an actual system as a binary sequence. The optimal perfect binary sequence is only $Z(4, 1, 1)$. The proposed sequence set can realize a flexible design of application systems. ZCZ width $z \leq 3$ is sufficient for usual applications.

For $m \geq 1$, the width of the ZCZ of the autocorrelation function of the proposed sequence set is exactly equal to the theoretical bound for the ZCZ sequence set, from $-(2^{m+1}-1)$ to $(2^{m+1}-1)$, whereas the width of the ZCZ of the cross-correlation function of the proposed sequence is equal to the theoretical upper bound, from $-(2^{m-1}3)$ to $(2^{m-1}3)$.

The simulation results of the application to an approximately synchronized CDMA (AS-CDMA) system or quasi CDMA (QS-CDMA) system show the high performance of the proposed sequence set.

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