

# Equations of Dynamics: Using the First Integrals in Numerical Integration

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## ABSTRACT

Equations of dynamics have often the first integrals. In particular, the canonical equations with stationary Hamiltonian have the energy integral. This fact is sometimes used to monitor the accuracy of numerical integration or to construct special methods of numerical integration, focused on the use of such equations.

In this paper we propose a method for correcting the numerical integration at each step based on the known first integrals of these equations.

For the numerical experiments we use three programs: DOP853 (the Dorman-Prince method), ODEX (the Gregg-Bulirsch-Stoer method) and TSMR (explicit method of Taylor series). Programs have been modified so that at each step of numerical integration the corrections ensure the constancy of the first integrals. Numerical experiments show that the proposed method sometimes allows to improve the accuracy of the results on long time intervals. Experiments were carried out on the Two-body problem and the N-body problem (Sun and five outer planets).

## Categories and Subject Descriptors

G.1.0 [Numerical analysis]: General

## General Terms

Theory

## Keywords

First integrals, conservative methods, differential equations

## 1. INTRODUCTION

This study is a logical continuation and propagation of the ideas described in article [8], where the property of conservation of first integrals has been investigated in numerical integration by various methods. The study also proposed a method that facilitates correction of coordinates in each

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step of integration to improve the integration accuracy. Numerical experiments were conducted on the two-body problem, while four first integrals (one energy integral and three area integrals) were used to correct the coordinates. It was observed that the area integrals are conserved much better than the energy integral. Therefore, we felt it logical to assume that the residual error in energy integral contributes essentially to correction of coordinates.

Besides, investigating only the energy integral allows to make the method more universal and to apply it to a broader class of problems. For example, stationary Hamilton's canonical equations have energy integral.

Based on the aforesaid, we tried to apply the described method using only the energy integral. For numerical experiments, we considered two options for the N-body problem. The first is when  $N = 2$  (Sun and Neptune) and the second is when  $N = 6$  (Sun and five outer planets). The last one is chosen as the most interesting for researchers [9].

## 2. PROBLEM STATEMENT

We consider a system of differential equations describing the motion of the Solar system planets in heliocentric coordinates [5].

$$\ddot{g}_{ij} = -\gamma(m_0 + m_i) \frac{g_{ij}}{r_{0i}^3} + \gamma \sum_{\substack{k \in [1:5] \\ k \neq i}} m_s \left[ \frac{g_{kj} - g_{ij}}{r_{ki}^3} - \frac{g_{ij}}{r_{0k}^3} \right], \quad (1)$$

where  $r_{ij}^2 = \sum_{j \in [1:3]} (g_{ij} - g_{kj})^2$ ,  $r_{si} > 0$ ,  $i \in [1 : 5]$ ,  $k \in [0 : 5]$ ,  $k \neq i$ ,  $j = 1, 2, 3$ ,  $\gamma$  — universal gravitational constant.

For the two-body problem, these equations take a simpler form [5]:

$$\ddot{g}_j = -\gamma(m_0 + m_1) \frac{g_j}{r^3}, \quad (2)$$

where  $r = \sqrt{g_1^2 + g_2^2 + g_3^2}$ .

Let us write the same equations in polynomial form. To do this, we use the additional variable method outlined in [3]. This conversion is done with the aim that the TSMR program [5] implementing the Taylor series method [4] is focused on solution of differential equations with polynomial right-hand sides.

$$\begin{cases} \dot{g}_{ij} = p_{ij}, \\ \dot{p}_{ij} = -\gamma(m_0 + m_i)g_{ij}z_{0i}^3 + \\ + \gamma \sum_{\substack{k \in [1:5] \\ k \neq i}} m_k \left[ (g_{kj} - g_{ij})z_{ki}^3 - g_{ij}z_{0k}^3 \right], \\ \dot{z}_{si} = -z_{si}^3 \sum_{j \in [1:3]} (g_{ij} - g_{sj})(p_{ij} - p_{sj}), \end{cases} \quad (3)$$

where  $i \in [1 : 5]$ ,  $s \in [0 : 5]$ ,  $s < i$ ,  $z_{si} = 1/r_{si}$  [5].

$$\begin{cases} \dot{g}_i = p_i, \\ \dot{p}_i = -\mu g_i z^3, \\ \dot{z} = -z^3(g_1 p_1 + g_2 p_2 + g_3 p_3), \end{cases} \quad i = \overline{1, 3}, \quad (4)$$

where  $\mu = \gamma(m_0 + m_1)$ ,  $z = 1/r$  and  $\gamma$  – universal gravitational constant [5].

Apart from the TSMR program mentioned above, we also used the DOP853 [1] and ODEX [1] programs in our numerical experiments. These two programs implement the Dormand–Prince [7] and Gragg–Bulirsch–Stoer [7] algorithms respectively.

The initial data used in the problems are presented in Table 1. The coordinates are presented in astronomical units

**Table 1: Coordinates, velocities and masses of outer Solar System planets**

Planet	$g_{i1}, g_{i2}, g_{i3}$	$p_{i1}, p_{i2}, p_{i3}$
Jupiter 1047.3486	$0.36484424231671 \times 10^1$	$0.51458739160244 \times 10^{-2}$
	$-0.31885628561843 \times 10^1$	$0.53770578567746 \times 10^{-2}$
	$-0.14570594138551 \times 10^1$	$0.21809890337032 \times 10^{-2}$
Saturn 3497.898	$0.860812008268 \times 10^{-1}$	$-0.58771844012993 \times 10^{-2}$
	$0.83323915033502 \times 10^1$	$-0.48077953809 \times 10^{-4}$
	$0.34416852468077 \times 10^1$	$0.02344003625752 \times 10^{-2}$
Uranus 22902.98	$-0.16894580321104 \times 10^2$	$0.15371000247255 \times 10^{-2}$
	$-0.6802790274465 \times 10^1$	$-0.3460060959201 \times 10^{-2}$
	$-0.2742015202544 \times 10^{-1}$	$-0.15378414591329 \times 10^{-2}$
Neptune 19412.24	$-0.1196586398024 \times 10^2$	$0.28675086989466 \times 10^{-2}$
	$-0.25873934395511 \times 10^2$	$-0.10995281783636 \times 10^{-2}$
	$-0.10297822028119 \times 10^2$	$-0.5224555223166 \times 10^{-3}$
Pluto $1.35 \times 10^8$	$-0.29628255336199 \times 10^2$	$0.747289767668 \times 10^{-3}$
	$-0.5542237950558 \times 10^1$	$-0.30948199515322 \times 10^{-2}$
	$0.7229040993881 \times 10^1$	$-0.12025228562757 \times 10^{-2}$

(AU), while the speed is presented in astronomical units per day ( $\text{AUd}^{-1}$ ) (mean solar day). To make computations easier the masses of the planets relative to the Sun are used in the programs instead of the true masses.

It is assumed that the initial data recorded in Table 1 are defined absolutely accurate.

We write the analytical formulas of energy integrals. Energy integral in the two-body problem [6]:

$$H(g, p) = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) - \frac{\mu}{\sqrt{g_1^2 + g_2^2 + g_3^2}}. \quad (5)$$

Energy integral in the N-body problem [6]:

$$\begin{aligned} H(g, p) = & \frac{1}{2} \sum_{i=1}^{n-1} m_i (p_{i1}^2 + p_{i2}^2 + p_{i3}^2) - \frac{1}{2m} \left[ \left( \sum_{i=1}^{n-1} m_i p_{i1} \right)^2 + \right. \\ & \left. + \left( \sum_{i=1}^{n-1} m_i p_{i2} \right)^2 + \left( \sum_{i=1}^{n-1} m_i p_{i3} \right)^2 \right] - \gamma \sum_{i < j, j \in [1:n-1]} \frac{m_i m_j}{r_{ij}}, \end{aligned} \quad (6)$$

where  $m = \sum_{i=0}^n m_i$ ,  $r_{ij} = \sqrt{\sum_{j \in [1:3]} (g_{ij} - g_{kj})^2}$ .

### 3. INTEGRAL CONSERVATION METHOD

Below is a description of the method used to conserve the values of the first integrals over an entire integration interval. This method is based on the method described in [8] view of the fact that we try to conserve only the energy integral.

Let  $x_0 = (g_0, p_0)$  be the initial data of the Cauchy problem for the system (1) or (2). Let  $x = (g, p)$ ,  $\dim(g, p) = n$  be the value on the  $k$ -th step of integration. The function  $H(x(t)) = C_0 = \text{const}$  is the integral energy of the corresponding system.

Let us denote by  $C_1 = H(x_1)$  the energy integral values in the  $k$ -th step. We decompose the integral in a Taylor series by discarding all the higher order terms

$$H(x + \delta) = C_{i0} = H(x) + \sum_{k=1}^n \frac{\partial H(x)}{\partial x_k} \delta_k.$$

Introducing the denotation  $d = C_0 - C_1$ , we get:

$$\sum_{k=1}^n \frac{\partial H(x)}{\partial x_k} \delta_k = d.$$

Let's seek  $\delta_1, \dots, \delta_n$  such that  $\sum_{k=1}^n \delta_k^2 \rightarrow \min$ . To find them, we will use the method of Lagrange multipliers:

$$L(\delta, \lambda) = \sum_{i=1}^n \delta_i^2 + \lambda \left( \sum_{k=i}^n \frac{\partial H(x)}{\partial x_i} \delta_i - d \right), \quad (7)$$

$$\frac{\partial L(\delta, \lambda)}{\partial \delta_i} = 2\delta_i + \frac{\partial H(x)}{\partial x_i} \lambda = 0, \quad i = \overline{1, n}, \quad (8)$$

$$\frac{\partial L(\delta, \lambda)}{\partial \lambda} = \frac{\partial L(\delta, \lambda)}{\partial \lambda} = \sum_{k=i}^n \frac{\partial H(x)}{\partial x_i} \delta_i - d = 0. \quad (9)$$

From (8),  $\delta_i$  becomes:

$$\delta_i = -\frac{1}{2} \frac{\partial H(x)}{\partial x_i} \lambda = 0, \quad i = \overline{1, n}. \quad (10)$$

Substituting (10) in (9) we obtain:

$$\lambda = -\frac{2d}{\sum_{i=1}^n \left( \frac{\partial H(x)}{\partial x_i} \right)^2} = 0. \quad (11)$$

The final formula for computing the numerical value of corrections is as follows:

$$\delta_i = \frac{d \cdot \frac{\partial H(x)}{\partial x_i}}{\sum_{k=1}^n \left( \frac{\partial H(x)}{\partial x_k} \right)^2}. \quad (12)$$

### 4. NUMERICAL EXPERIMENTS

Taking into account the method described above, we modified all the programs in such a way as to correct the solution values using this method at the end of each period. These modifications are presented in Fig. 1

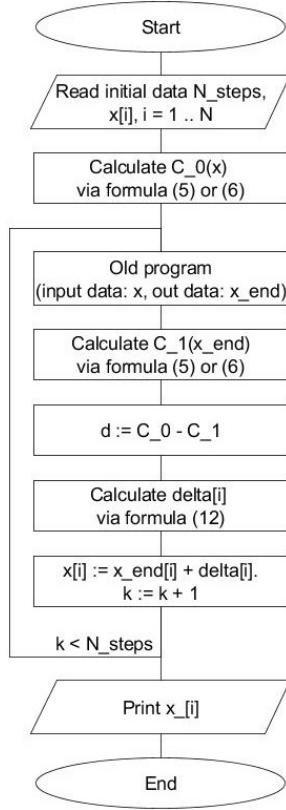


Figure 1: Block diagram of re-engineered program

#### 4.1 Two-body problem

The numerical experiment results are presented in Fig. 2.

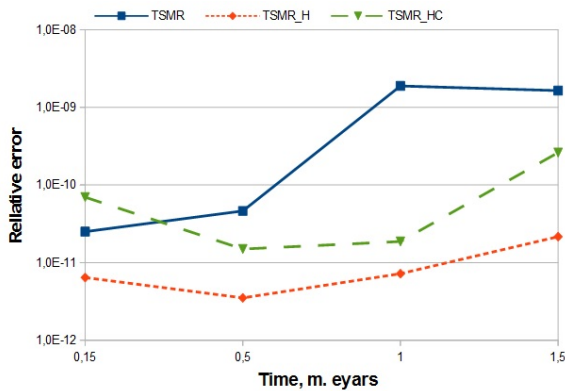


Figure 2: Results of integration using the TSMR program

The figure shows the results of integration carried out using the TSMR program and its modifications. TSMR-HC is

the modification of TSMR and it conserves the energy integral and the area integral, while TSMR-H is the modification that conserves only the energy integral.

It can be seen in the graph that among all the modifications, the TSMR-H program yields the best results (1-2 orders of magnitude). One can note that this result is remarkable also by the fact that integration accuracy improved despite reducing the amount of integrals conserved. Besides, as has been noted above, this simplifies the program itself and makes it more versatile.

The DOP853 and ODEX programs give similar results. Detailed graphs and tables for a variety of programs can be found in [2].

#### 4.2 Problem of movement of outer Solar System planets

Given the results in the preceding paragraph, we have attempted to test our program on the six-body problem. Here, we decided to use the TSMR-H modification that conserves only the energy integral. The results obtained were unpleasantly surprising.

Since the N-body problem has no periodic solution for  $N > 2$ , round-trip integration was therefore used to assess the relative solution errors: the equations are first integrated from point 0 to point  $T$ , and then, taking the data obtained as initial data, the equations are integrated from point  $T$  to point 0.

As shown by experiments on all the tested intervals, the integration accuracy worsens by several orders of magnitude. After obtaining these results, we decided to check how the modified program behaves in one step for round-trip integration.

The results are shown in Table 2.

Table 2: Result of integration in one step

TSMR	TSMR-H	TSMR-H-NEW
$1.597274 \times 10^{-12}$	$6.743340015638 \times 10^{-3}$	$6.743340015644 \times 10^{-3}$

The Table 2 contains errors for integration using TSMR and its two modifications. The TSMR-H modification differs from TSMR-H-NEW with the fact that in the TSMR-H program, the initial value of energy integral that we conserve further is calculated once at the starting point of integration and is used in the entire round-trip integration area. In the TSMR-H-NEW program, the initial value is calculated twice – the first time is at the starting point of integration when integrating forward and the second time is at the starting point of integration when integrating back.

As it follows from the table, although the values of the first integrals are conserved in both cases, integration accuracy does not improve but worsens.

### 5. CONCLUSIONS

The paper describes an integration method that allows to conserve the energy integral value in the entire integration interval. This method was used to modify some existing programs and conduct numerical experiments.

In the two-body problem, using the proposed method can improve integration accuracy by 1 – 2 orders of magnitude.

In the problem of five outer Solar System planets, applying the method reduces the integration accuracy. Character of

result degradation is not clear yet. We can note the following reasons:

- instability of the 6-body problem. In other words, a small change of initial data influences on further motion significantly.
- unintentional mistake in the program modification. Although there is not any explicit or indirect confirmation of this fact (energy integrals are saved with high accuracy, modification does not influence on main integration functions Fig.1), we do not eliminate this possibility.

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