On the Multifacility Weber Problem for Four Terminals

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ABSTRACT

We study a generalization of the Euclidean minimal tree problem to the case of the planar weighted networks consisting of four given terminals and two extra facilities. Explicit analytical formulae are presented for the conditions of the existence of the network, facility coordinates and for the total network cost. These formulae are utilized for the investigation of the network dynamics under variation of parameters.

Categories and Subject Descriptors

G.2.2 [**Discrete mathematics**]: Graph Theory—*Network problems*; G.1.6 [**Numerical analysis**]: Optimization—*Global optimization*; I.1.2 [**Symbolic and algebraic manipulation**]: Algorithms—*Algebraic Algorithms*

General Terms

Theory, Algorithms

Keywords

Euclidean multifacility location problem, Weber problem, nonlinear optimization, analytical solution

1. INTRODUCTION

The classical *Weber* or *generalized Fermat-Torricelli problem* is stated as that of finding the point (facility, junction) $S = (x_*, y_*)$ that minimizes the sum of weighted distances from itself to $n \ge 3$ fixed points (terminals) $\{P_j = (x_j, y_j)\}_{j=1}^n$ in the Euclidean plane:

$$\min_{S\in\mathbb{R}^2} \sum_{j=1}^{n} m_j |SP_j| \, . \qquad (1)$$

Hereinafter $|\cdot|$ stands for the Euclidean distance and the weights $\{m_j\}_{j=1}^n$ are assumed to be positive real numbers.

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This nonlinear optimization problem was stated by Alfred Weber [6] with regard to the optimal facility location problem like the one of finding the optimal, i.e., minimizing the transportation costs, position for the plant manufacturing one ton of the final product from $\{m_j\}_{j=1}^n$ tons of distinct raw materials located at $\{P_j\}_{j=1}^n$. He also treated the *multifacility problem* consisting in finding the set of $\ell \ge 2$ facility points $\{S_i\}_{i=1}^{\ell}$ in $\mathbb{R}^2$ connected to the terminals $\{P_j\}_{j=1}^n$ that solves the optimization problem

$$\min_{\{S_1,\dots,S_\ell\}\subset\mathbb{R}^2} \left\{ \sum_{j=1}^{n}\sum_{i=1}^{\ell} m_{ij}|S_iP_j| + \sum_{k=1}^{\ell}\sum_{i=k+1}^{\ell-1} \widetilde{m}_{ik}|S_iS_k| \right\} .$$

This problem can be considered as a natural generalization of the celebrated *Steiner minimal tree problem* aimed at construction of the network of the minimal length linking the given terminals.

Dozens of papers are devoted to the Weber problem, its ramifications and applications; we refer to [2] and [3] for the reviews. They are focused onto the application of the variety of numerical procedures for the nonlinear optimization problem. The main difficulty consists in the fact that the objective (or cost) function of the Weber problem is nondifferentiable, and the extensions of the standard nonlinear programming versions of iterative procedures for finding its minimum, like the gradient descent ones, should be modified. One of these modifications is based on the Weiszfeld algorithm.

The present paper is devoted to an alternative approach for the problem, namely an analytical one. We are looking for the explicit expressions for the facility coordinates as functions of the problem parameters (terminal coordinates and weights). This approach has been originated in the recent papers [4] and [5] where the unifacility Weber problem for three terminals and Steiner minimal tree problem for four terminals have been solved *by radicals*. Within the framework of this approach, we will be focused here on solution of the Weber problem for the case of $n = 4$ terminals and $\ell = 2$ facilities, namely we are looking for the facilities $S_1 = (x_*, y_*)$ and $S_2 = (x_{**}, y_{**})$ minimizing the following objective function

$$F(x_*, y_*, x_{**}, y_{**}) = m_1|S_1P_1| + m_2|S_1P_2| + m_3|S_2P_3| + m_4|S_2P_4| + m|S_1S_2| \, . \qquad (2)$$

We prove that this problem can also be solved by radicals with the main result of paper formulated in the next section.

2. ANALYTICS

We will treat the case where the terminals $\{P_j\}_{j=1}^4$, while counted counterclockwise, compose a convex quadrilateral $P_1P_2P_3P_4$.

THEOREM 1. *The necessary condition for the existence of solution to the Weber problem is that of positivity of the values*

$$k_{12} = (m+m_1+m_2)(m-m_1+m_2) \times (m+m_1-m_2)(-m+m_1+m_2),$$

$$k_{34} = (m+m_3+m_4)(m-m_3+m_4) \times (m+m_3-m_4)(-m+m_3+m_4).$$

Set

$$\tau_1 = \sqrt{k_{12}}[\sqrt{k_{34}}(x_4-x_3) - (m^2+m_3^2-m_4^2)y_3 - (m^2-m_3^2+m_4^2)y_4] + 2m^2\sqrt{k_{12}}y_2 + k_{12}(x_1-x_2) + (m^2+m_1^2-m_2^2)[\sqrt{k_{34}}(y_3-y_4) + (m^2+m_1^2-m_2^2)x_1 + (m^2-m_1^2+m_2^2)x_2 - (m^2+m_3^2-m_4^2)x_3 - (m^2-m_3^2+m_4^2)x_4],$$

$$\tau_2 = -\sqrt{k_{12}}[\sqrt{k_{34}}(x_4-x_3) - (m^2+m_3^2-m_4^2)y_3 - (m^2-m_3^2+m_4^2)y_4] - 2m^2\sqrt{k_{12}}y_1 - k_{12}(x_1-x_2) + (m^2-m_1^2+m_2^2)[\sqrt{k_{34}}(y_3-y_4) + (m^2+m_1^2-m_2^2)x_1 + (m^2-m_1^2+m_2^2)x_2 - (m^2+m_3^2-m_4^2)x_3 - (m^2-m_3^2+m_4^2)x_4],$$

$$\eta_1 = \sqrt{k_{12}}[\sqrt{k_{34}}(y_4-y_3) + (m^2+m_3^2-m_4^2)x_3 + (m^2-m_3^2+m_4^2)x_4] - 2m^2\sqrt{k_{12}}x_2 + k_{12}(y_1-y_2) + (m^2+m_1^2-m_2^2)[\sqrt{k_{34}}(x_4-x_3) + (m^2+m_1^2-m_2^2)y_1 + (m^2-m_1^2+m_2^2)y_2 - (m^2+m_3^2-m_4^2)y_3 - (m^2-m_3^2+m_4^2)y_4],$$

$$\eta_2 = -\sqrt{k_{12}}[\sqrt{k_{34}}(y_4-y_3) + (m^2+m_3^2-m_4^2)x_3 + (m^2-m_3^2+m_4^2)x_4] + 2m^2\sqrt{k_{12}}x_1 - k_{12}(y_1-y_2) + (m^2-m_1^2+m_2^2)[\sqrt{k_{34}}(x_4-x_3) + (m^2+m_1^2-m_2^2)y_1 + (m^2-m_1^2+m_2^2)y_2 - (m^2+m_3^2-m_4^2)y_3 - (m^2-m_3^2+m_4^2)y_4],$$

and set the values for $\tau_3, \tau_4, \eta_3, \eta_4$ via the formulae obtained by cyclic substitution for subscripts

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

in the above expressions for $\tau_1, \tau_2, \eta_1, \eta_2$ correspondingly.
If all the values

$$\delta_1 = \eta_2(x_1-x_2) + \tau_2(y_2-y_1),$$

$$\delta_2 = \eta_1(x_1-x_2) + \tau_1(y_2-y_1),$$

$$\delta_3 = \eta_4(x_3-x_4) + \tau_4(y_4-y_3),$$

$$\delta_4 = \eta_3(x_3-x_4) + \tau_3(y_4-y_3),$$

$$\delta = -\frac{\delta_1\left(m^2+m_1^2-m_2^2\right)}{\sqrt{k_{12}}} - \frac{\delta_3\left(m^2+m_3^2-m_4^2\right)}{\sqrt{k_{34}}} + (\eta_1+\eta_2)(y_1-y_3) + (\tau_1+\tau_2)(x_1-x_3)$$

are positive then there exists a pair of points S_1 and S_2 lying inside $P_1P_2P_3P_4$ that furnishes the minimal value for (2).
The coordinates of point S_1 are as follows:

$$x_* = x_1 - \frac{2\delta_1 m^2 \tau_1}{\sqrt{k_{34}}\left[(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2\right]}, \qquad (3)$$

$$y_* = y_1 - \frac{2\delta_1 m^2 \eta_1}{\sqrt{k_{34}}\left[(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2\right]} \qquad (4)$$

while those of point S_2:

$$x_{**} = x_3 - \frac{2\delta_3 m^2 \tau_3}{\sqrt{k_{12}}\left[(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2\right]}, \qquad (5)$$

$$y_{**} = y_3 - \frac{2\delta_3 m^2 \eta_3}{\sqrt{k_{12}}\left[(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2\right]}. \qquad (6)$$

The minimal value for (2) *(cost of the network) then equals*

$$\mathfrak{C} = \frac{\sqrt{(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2}}{4m^3}. \qquad (7)$$

Theorem 1 claims that, for the case of two facilities, the Weber problem can be solved by radicals. The proof is similar to its counterpart for the equal weighted case [5]. It can be proved that the $\{4 \text{ terminals}, 2 \text{ facilities}\}$-Weber problem can be reduced to a twain of $\{3 \text{ terminals}, 1 \text{ facility}\}$-Weber problems. For instance, the configuration of weights

$$\{P_1, m_1\}, \{P_2, m_2\}, \{Q, m\}$$

with

$$Q = 1/(2m^2)$$

$$\times \ \Big(m^2(x_3+x_4) + (m_3^2-m_4^2)(x_3-x_4) - \sqrt{k_{34}}(y_3-y_4), \ m^2(y_3+y_4) + (m_3^2-m_4^2)(y_3-y_4) + \sqrt{k_{34}}(x_3-x_4)\Big)$$

possesses a solution to the unifacility problem coinciding with the position of the facility S_1. For this type of problems, an analytical solution is already constructed [4].

COROLLARY 1. *Under the conditions of Theorem 1, the point $S_1 = (x_*, y_*)$ lies inside the triangle $P_1P_2S_2$ and provides a solution to the unifacility Weber problem*

$$\min_{S\in\mathbb{R}^2}\left(m_1|SP_1| + m_2|SP_2| + m|SS_2|\right).$$

Similar statement is also valid for the point $S_2 = (x_{**}, y_{**})$ and the terminals P_3, P_4 and S_1.

We outline briefly the meaning of the conditions from Theorem 1. First, due to Heron's formula, the values $\frac{1}{4}\sqrt{k_{12}}$ and $\frac{1}{4}\sqrt{k_{34}}$ equal the squares of the so-called weight triangles, i. e. the triangles composed with the sets of edges coinciding with $\{m_1, m_2, m\}$ and $\{m_3, m_4, m\}$ respectively. The positivity of k_{12} and k_{34} guarantees the existence of both weight triangles. The condition for the positivity of all the delta values from the statement of Theorem 1 is essential for the problem solubility. Conditions $\{\delta_j > 0\}_{j=1}^4$ ensure the location of the facilities S_1 and S_2 inside the quadrilateral $P_1P_2P_3P_4$, while the condition $\delta > 0$ guarantees the facilities against their collision since

$$|S_1S_2| = \frac{\delta}{\sqrt{(\eta_1+\eta_2)^2 + (\tau_1+\tau_2)^2}}. \qquad (8)$$

Corollary 2. *For the equal weighted case* $\{m_j = 1\}_{j=1}^4$, $m = 1$, *the expression for* δ *can be represented in the form*

$$\delta = \frac{8}{\sqrt{3}} [x_3 - x_1, y_3 - y_1] \cdot \left[\begin{array}{cc} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{array} \right] \cdot \left[\begin{array}{c} x_4 - x_2 \\ y_4 - y_2 \end{array} \right] .$$

This value is positive iff the angle between the diagonal $\overrightarrow{P_1P_3}$ *of the quadrilateral and the other diagonal* $\overrightarrow{P_2P_4}$ *turned through by* $\pi/6$ *clockwise is acute. Equivalently, if we denote by* ψ *the angle between diagonal vectors* $\overrightarrow{P_1P_3}$ *and* $\overrightarrow{P_2P_4}$ *then* δ *is positive iff* $\psi < \pi/2 + \pi/6 = 2\pi/3$. *This confirms the known condition for the existence of a full Steiner tree of the following topology:*

$$\begin{array}{c} P_1 \\ P_2 \end{array} S_1 S_2 \begin{array}{c} P_4 \\ P_3 \end{array} .$$

Formulae (3)-(6) yield then the coordinates of Steiner points while the length of the corresponding Steiner tree equals

$$\mathfrak{C} = \frac{1}{2}\sqrt{A^2 + B^2}$$

where

$$\begin{array}{rcl} A & = & \sqrt{3}(x_1 - x_2 - x_3 + x_4) + (y_1 + y_2 - y_3 - y_4), \\ B & = & (x_1 + x_2 - x_3 - x_4) + \sqrt{3}(-y_1 + y_2 + y_3 - y_4) . \end{array}$$

3. EXAMPLES

In comparison with numerical (iterative) procedures for solving the Weber problem, representation of its solution in analytical form given in the previous section looks cumbersome. However, we give the following reasons for its utility:

- Though the numerical procedures are generically faster if dealing with particular specialization of the problem parameters, their approximation properties might be invalid when at least one of facilities being searched happens to lie close to a terminal position. On the contrary, analytical formulae are universal in the sense that they yield the exact result (i.e., free of truncation errors) regardless on the position of facilities.
- In case of the problem dealing with some variable parameters, analytics provide one with a unique opportunity to evaluate their influence on its solution. In particular, this means that the *bifurcation values* for the parameters can be determined responsible for the degeneracy of the network topology.

In the present section we will exemplify the latter point.

Example 1. *Find the coordinates of facilities* S_1, S_2 *for the following configuration of weights*

$$\left. \begin{array}{c} P_1 = (1,5) \\ m_1 = 3 \end{array} \right| \begin{array}{c} P_2 = (2,1) \\ m_2 = 2 \end{array} \left| \begin{array}{c} P_3 = (7,2) \\ m_3 = 3 \end{array} \right| \begin{array}{c} P_4 = (6,7) \\ m_4 = 4 \end{array} \Bigg| \, m = 4 \, .$$

Solution. The conditions of Theorem 1 are fulfilled: the values $\delta_1 \approx 14124, \delta_2 \approx 29388, \delta_3 \approx 34784, \delta_4 \approx 18831$ and $\delta \approx 11721$ are positive. Formulae (3)-(7) then give the coordinates for facilities

$$x_* = \frac{2266800 + 772027\sqrt{15} + 453552\sqrt{33} + 246177\sqrt{55}}{48\left(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55}\right)} \approx 3.701271,$$

$$y_* = \frac{1379951 + 201984\sqrt{15} + 97279\sqrt{33} + 154368\sqrt{55}}{16\left(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55}\right)} \approx 4.430843,$$

$$x_{**} \approx 4.761622, \quad y_{**} \approx 4.756175,$$

and the cost of the network (Fig. 1):

$$\mathfrak{C} = \frac{1}{8}\sqrt{44098 + 4170\sqrt{15} + 5118\sqrt{55} + 1890\sqrt{33}} \approx 41.280608.$$

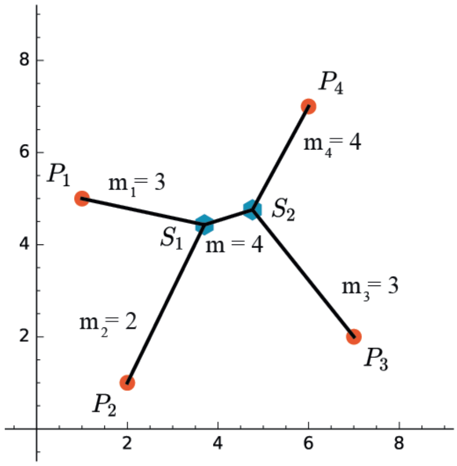

Figure 1: Network for the configuration of weights from Example 1

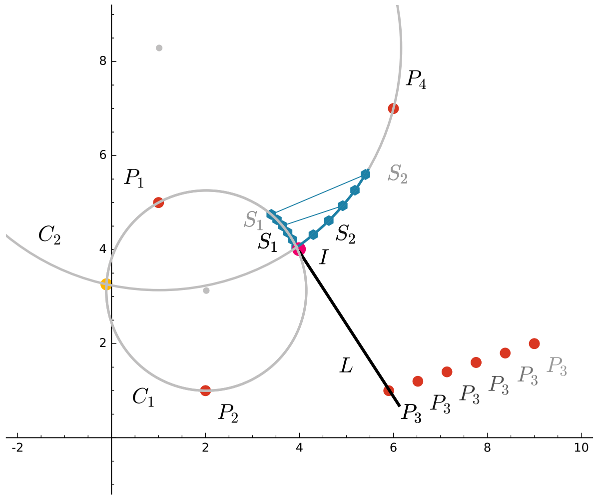

Figure 2: Dynamics of points S_1, S_2 under variation of terminal P_3.

Example 2. *For the terminals* P_1, P_2, P_4 *from Example 1 and for* P_3 *moving towards* P_2 *from the starting position at* $(9,2)$ *find the loci of facilities* S_1, S_2.

Solution. It turns out that when P_3 wanders, the facility S_j moves along the arc of the circle

$$C_j = \left\{ (x,y) \in \mathbb{R}^2 \middle| \, (x - X_j)^2 + (y - Y_j)^2 = r_j^2 \right\} .$$

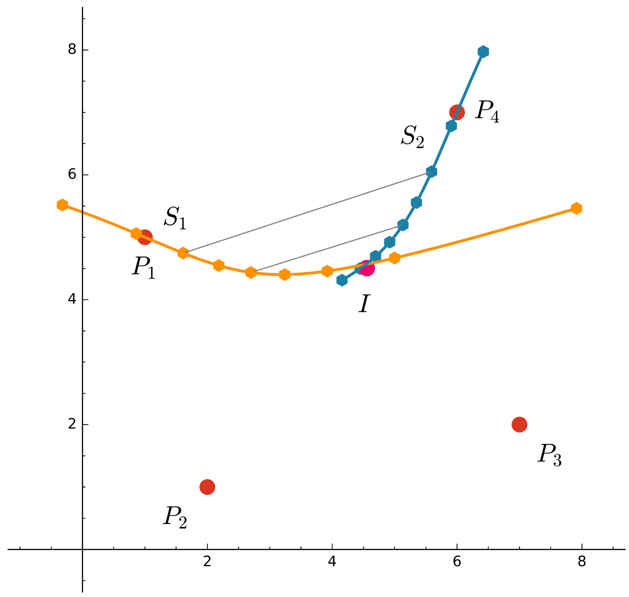

Figure 3: Dynamics of points S_1, S_2 under variation of weight m.

Here

$$X_1 = \frac{1}{30}(45+4\sqrt{15}), Y_1 = \frac{1}{30}(90+\sqrt{15}), r_1 = \frac{2}{15}\sqrt{255} ;$$

while the exact expressions for the parameters of C_2 are rather complicated and we present here just only their approximations:

$$X_2 \approx 1.013521, \ Y_2 \approx 8.288416, \ r_2 \approx 5.150241 .$$

We emphasize that the trajectory of P_3 does not influence the trajectories of S_1 and S_2, i.e. both facilities do not leave the corresponding arcs for any drive of P_3 until the latter reaches the line

$$L \approx \left\{ (x,y) \in \mathbb{R}^2 \Big| \ y = -1.538431\,x + 10.104975 \right\} .$$

At this moment, S_1 coincides with S_2 in the point

$$I \approx (3.936925, 4.048287)$$

which yields a solution for the unifacility Weber problem (1) for the terminals $\{P_j\}_{j=1}^4$. Point I is invariant for any position of P_3 in L (Fig. 2).

The scenario for the facilities behaviour in the present example looks similar to the equal weighted case [5], while the problem statement of the next example is of a completely new nature.

Example 3. *For the terminals $\{P_j\}_{j=1}^4$ from Example 1 find the loci of facilities S_1, S_2 under the variation of the weight m within* $[2, 4.8]$.

Solution. When the weight m increases, the facilities S_1 and S_2 approach each other along the algebraic curves given in parametric form as $(x_*(m), y_*(m))$ and $(x_{**}(m), y_{**}(m))$ correspondingly. Due to (8), these points collide when m coincides with a zero of the equation $\delta(m) = 0$. The latter can be reduced to an algebraic one

$$24505\,m^{20} - 3675750\,m^{18} + \cdots + 25596924755077 = 0$$

with a zero $m_0 \approx 4.326092$. The collision point

$$I \approx (4.537574, 4.565962)$$

yields a solution for the unifacility Weber problem (1) for the terminals $\{P_j\}_{j=1}^4$.

When m decreases, the facility S_1 moves towards P_1 while S_2 moves towards P_4. The first drive is faster than the second one: S_1 approaches P_1 when m coincides with a zero of the equation $\delta_1(m) = 0$. The latter can be reduced to an algebraic one

$$377145\,m^{12} - 15186678\,m^{10} + \cdots + 8631109474 = 0$$

with a zero $m_1 \approx 2.405703$ (Fig. 3).

4. CONCLUSIONS

The multifacility Weber problem for the case of four terminals and two facilities in the plane has been tackled in its general statement including establishment the conditions for its solubility and deduction the explicit formulae for its solution. The obtained result permits one to analyze the effect of the problem parameter variations to the shape of the network. It also inspires a hope in extensibility of the analytical approach to the problem in its general statement. Indeed, on recalling the idea underlying the proof of Theorem 1, one might expect that the general n-terminal Weber problem can be somehow reduced to a couple of $(n-1)$-terminal problems. Right at the moment, this statement is a mere conjecture, however it is also justified by the known in the literature treatment of the particular case of the problem, namely the Steiner minimal tree problem.

The result can also be useful for the data clusterization problems and for the phylogenetic tree reconstruction [1]. For the latter problem, the extension of the results of present paper to $\mathbb{R}^d, d \ge 3$ is a question for further investigation.

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5. REFERENCES

[1] M. Brazil et al. A novel approach to phylogenetic trees: d-dimensional geometric Steiner trees. *Networks*, 53(2):104–111, 2009.

[2] Z. Drezner et al. The Weber problem. *Springer-Verlag*, Facility Location: Applications and Theory:1–36, 2002.

[3] C. ReVelle and H. Eiselt. Location analysis: A synthesis and survey. *European Journal of Operational Research*, 165(1):1–19, 2005.

[4] A. Uteshev. Analytical solution for the generalized Fermat-Torricelli problem. *Amer. Math. Monthly*, 121(4):318–331, 2014.

[5] A. Uteshev. Some analytics for Steiner minimal tree problem for four terminals, 2015. arXiv:1505.03564v1.

[6] A. Weber. *The theory of the location of industries.* Chicago University Press, Chicago, 1929.