Is the Modern Theory of Stochastic Processes Complete? Example of Markovian Random Walks with Constant Non-Symmetric Diffusion Coefficients

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ABSTRACT

A new type non-symmetric diffusion problem is considered and the corresponding Brownian motion implementing such diffusion processes is constructed. As a particular example, *random walks with internal causality* on a square lattice are studied in detail. By construction, one elementary step of a random walker on the lattice may consist of its two succeeding jumps to the nearest neighboring nodes along the x- and then y-axis or the y- and then x-axis ordered, e.g., clockwise. It is essential that the second fragment of elementary step is caused by the first one, meaning that the second fragment can arise only if the first one has been implemented, but not vice versa. In particular, if for some reasons the second fragment is blocked, the first one may be not affected, whereas if the first fragment is blocked, the second one cannot be implemented in any case. As demonstrated, on time scales much larger then the duration of one elementary step these random walks are characterized by a diffusion matrix with non-zero anti-symmetric component. The existence of this anti-symmetric component is also justified by numerical simulation.

Categories and Subject Descriptors

G.3 [**Probability and Statistics**]: stochastic processes

General Terms

Theory

Keywords

Stochastic process, diffusion matrix, boundary conditions

1. INTRODUCTION

The present paper poses a fundamental question about the completeness of the modern formalism of describing stochastic processes and, by way of example, the formalism of the Fokker-Planck equations, or speaking more strictly, the forward Fokker-Planck equations is analyzed.

The Fokker-Planck equation (see, e.g., [1])

$$\partial_t G = \sum_{i=1}^{N} \partial_i \left\{ \sum_{j=1}^{N} \partial_j [D_{ij}(\mathbf{x},t)G] - V_i(\mathbf{x},t)G \right\} \qquad (1)$$

subject to the initial condition

$$G(\mathbf{x},t|\mathbf{x}_0,t_0)\big|_{t=t_0} = \delta(\mathbf{x}-\mathbf{x}_0)\,, \qquad (2)$$

where $\mathbf{x} = \{x_i\}_{i=1}^{i=N} \in \mathbb{Q} \subset \mathbb{R}^N$ and $t > t_0$, describes a wide class of Markovian random walks continuous in space and time for which the first and second moments of walker displacement are some finite space-continuous quantities. The matrix $\mathbf{D} = \|D_{ij}\|$ of diffusion coefficients and the velocity drift $\mathbf{V} = \{V_i\}$ in the "phase" space $\mathbb{R}^N$ are introduced as

$$D_{ij}(\mathbf{x},t) = \lim_{\tau\to 0} \frac{1}{2\tau} \left\langle (x'_i - x_i)(x'_j - x_j) \right\rangle_{x':(t+\tau|\mathbf{x},t)}\,, \qquad (3)$$

$$V_i(\mathbf{x},t) = \lim_{\tau\to 0} \frac{1}{\tau} \left\langle (x'_i - x_i) \right\rangle_{x':(t+\tau|\mathbf{x},t)}\,. \qquad (4)$$

Due to the form of the Fokker-Planck equation the diffusion coefficient matrix $\|D_{ij}\|$ must be symmetric, $D_{ij} = D_{ji}$, which follows from definition (3) as well.

Discrete random walks on lattices also admit this description on scales $t \gg \tau$, where τ is the characteristic time of the walker hopping to the neighboring lattice nodes. An example of symmetric (i.e. without regular drift, $\mathbf{V} = 0$) random walks on a square lattice is illustrated in Fig. 1: "diagram of transitions." Within one elementary time step τ a walker hops to one of the nearest lattice nodes with the probability $p = \frac{1}{4}(1-\epsilon)$ or to one of the next shell of nearest neighbors with the probability $q = \frac{1}{4}\epsilon$, here $0 < \epsilon < 1$ is a given parameter. For these random walks the diffusion matrix is of the diagonal form and can be characterized by one diffusion coefficient $D = (1+\epsilon)a^2/(4\tau)$, i.e., $D_{xx} = D_{yy} = D$ and $D_{xy} = D_{yx} = 0$.

Appealing to the form of the Fokker-Planck equation (1) usually one draws a conclusion that the diffusion flux $\mathbf{J} = \{J_i\}$ is related to the distribution function G via the expression

$$J_i = -\sum_{j=1}^{N} \partial_j [D_{ij}(\mathbf{x},t)G] + V_i(\mathbf{x},t)G\,. \qquad (5)$$

Then ascribing various physical properties to the medium boundary $\partial\mathbb{Q}$ the Fokker-Planck equation is subjected to

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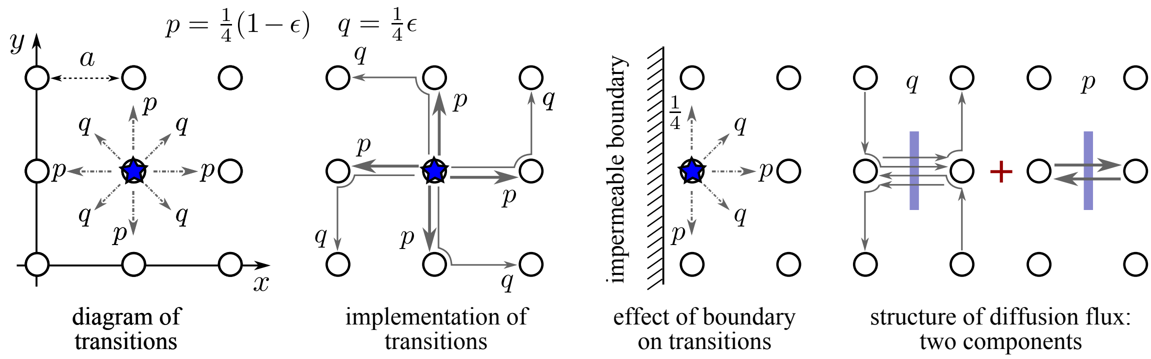

Figure 1: The analyzed random walks on the square lattice, from left to right, the diagram showing possible transitions of the walker within one elementary step and their probabilistic weights, spatial structure of these transitions, diagram and probabilistic weights of the walker near a impermeable boundary, the diagram illustrating the relationship between the diffusion flux and possible walker transitions.

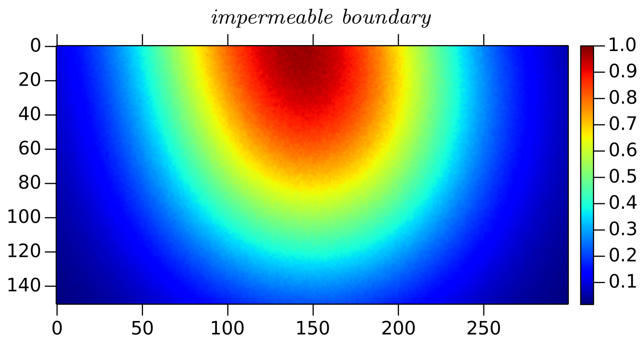

Figure 2: Distribution function normalized to its maximum. In numerical simulation the asymmetry parameter $\epsilon = 0.2$, the trajectory origin $\{x_0 = 150, y_0 = 50\}$, and the number of steps in one trajectory $N = 3y_0^2$ were used.

the corresponding boundary conditions (see, e.g., [1]). The latter completes the description of such stochastic processes in the framework of the Fokker-Planck equation. If it were so for the random walks illustrated in Fig. 1, at a impermeable boundary (third fragment counting from left in Fig. 1) the distribution function G would meet the boundary condition $\partial_x G = 0$ in the continuous approximation.

As far as the relationship between the Fokker-Planck equation (1) and the diffusion flux (5) is concerned, we note that the replacement $D_{ij} \Longrightarrow D_{ij} + D^a_{ij}$, where D^a_{ij} is an asymmetric component, $D^a_{ij} = -D^a_{ji}$, does not change the form of the Fokker-Planck equation (1) but modifies substantially expression (5). The latter, in turn, changes the boundary condition, so, finally, contributes essentially to the description of stochastic process. Therefore the statement on the diffusion coefficient symmetry does not follow from the derivation of the Fokker-Planck equation but is an *addition assumption* that can be accepted for some physical reasons.

2. MODEL AND DISCUSSION

The example of random walks shown in Fig. 1 illustrates the stated above proposition, in particular, the diffusion coefficient matrix D_{ij} entering relationship (5) is of the form containing antisymmetric component, namely,

$$D_{xx} = D_{yy} = \frac{(1+\epsilon)a^2}{4\tau}\,, \quad D_{xy} = -D_{yx} = -\frac{\epsilon a^2}{2\tau}\,. \qquad (6)$$

This fact must be reflected in the boundary conditions and, finally, cause the asymmetry of the distribution function for the diffusion problem in the region with the impermeable boundary with respect to the boundary point nearest to the origin of random walks. Numerical simulation justifies this statement (Fig. 2).

Concluding the obtained results, we pose a question about the completeness of describing stochastic processes in terms of the Fokker-Planck equation or stochastic differential equations. Indeed, this formalism ignores the *internal* structure of elementary steps, whereas the given example demonstrates the fact that particular spatial details of the walker motion within one elementary step can affect the macroscopic behavior of diffusion processes. Diffusion in magnetic field is also discussed as a characteristic example of physical systems where such phenomena can be pronounced. Besides it should be noted that the considered problem of non-symmetric diffusion coefficient matrix is partly related to the problem called non-symmetric diffusion dealing with a diffusion type stochastic processes governed by equations like

$$\partial_t G = \sum_{i,j=1}^{N} \partial_i \left[D_{ij}(\mathbf{x})\partial_j G\right]\,,$$

where the diffusion coefficient $D_{ij}(\mathbf{x})$ depends on the spatial coordinates $\mathbf{x}$ and, so, its possible non-symmetry can be responsible for macroscopic effects (see, e.g., [2] and references therein).

3. REFERENCES

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