Conditional and Joint Entropy

Conditional Entropy

$$H(Y|X=v)$$

Suppose I'm trying to predict output Y and I have input X

Conditional Entropy H(Y|X=v)

Example

If I know the student's major could I predict if he likes computer games?

Input: X = College Major

Output: Y = Likes “Computer Games”

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Marginal distribution for X

X \ Y	Yes	No	
Math	0.25	0.25	P(X=Math)=0.5
CS	0.25	0	P(X=CS)=0.25
History	0	0.25	P(X=History)=0.25
	P(Y=Yes)=0.5	P(Y=No)=0.5	1

Marginal distribution for Y

Conditional Entropy H(Y|X=v)

Example

If I know the student's major could I predict if he likes computer games?

Input: X = College Major

Output: Y = Likes "Computer Games"

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

X \ Y	Yes	No	
Math	0.25	0.25	P(X=Math)=0.5
CS	0.25	0	P(X=CS)=0.25
History	0	0.25	P(X=History)=0.25
	P(Y=Yes)=0.5	P(Y=No)=0.5	1

$H(X) = -0.5 \log 0.5 - 0.25 \log 0.25 - 0.25 \log 0.25 = 1.5$

$H(Y) = -0.5 \log 0.5 - 0.5 \log 0.5 = 1$

Specific Conditional Entropy H(Y|X=v)

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Specific Conditional Entropy H(Y|X=v)

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Example:

$H(Y|X=Math) = - S \quad p(y|X=Math) \log p(y|X=Math)$

y ? {Yes, No}

=-p(Yes|X=Math) log p(Yes|X=Math) -p(No|X=Math) log p(No|X=Math)

=-0.5 log 0.5 – 0.5 log 0.5 =1

H(Y|X=Math) = 1

Specific Conditional Entropy H(Y|X=v)

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Example:

$H(Y|X=History) = -S\ p(y|X=History) \log p(y|X=History)$

y ? {Yes, No}

=-p(Yes|X=History) log p(Yes|X=History) -p(No|X=History) log p(No|X=History)

=-0 log 0 – 1 log 1 =0

H(Y|X=History) = 0

Specific Conditional Entropy H(Y|X=v)

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Example:

$$H(Y|X=CS) = -S_{y ? \{Yes, No\}} p(y|X=CS) \log p(y|X=CS)$$

$= -p(Yes|X=CS) \log p(Yes|X=CS) - p(No|X=CS) \log p(No|X=CS)$

$= -1 \log 1 - 0 \log 0 = 0$

$$H(Y|X=CS) = 0$$

Specific Conditional Entropy $H(Y|X=v)$

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Specific Conditional Entropy:

$H(Y\,|X=v)$ = **The entropy of** Y **among only those records in which** X **has value** v

Example:

- $H(Y|X=Math) = 1$
- $H(Y|X=History) = 0$
- $H(Y|X=CS) = 0$

Conditional Entropy H(Y|X)

Is the amount of information contained in Y such that X is given

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Conditional Entropy:

$H(Y|X)$ = The average specific conditional entropy of Y

= if you choose a record at random what will be the conditional entropy of Y, conditioned on that row's value of X

= Expected number of bits to transmit Y if both sides will know the value of X

$= S_j \, Prob(X=v_j) \, H(Y \mid X = v_j)$

Conditional Entropy

Example

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

Definition of Conditional Entropy:

$H(Y|X)$ = The average conditional entropy of Y

$= S_j Prob(X=v_j)\ H(Y \mid X = v_j)$

Example:

v_j	$Prob(X=v_j)$	$H(Y \mid X = v_j)$
Math	0.5	1
History	0.25	0
CS	0.25	0

$H(Y|X) = 0.5 * 1 + 0.25 * 0 + 0.25 * 0 = 0.5$

Joint Entropy

Is the amount of information contained in both events X and Y

$$H(X, Y) = -S_{X,Y}\; p(x,y) \log p(x,y)$$

Joint Entropy

Chain Rule

Relationship between conditional and joint entropy

$$H(X,Y) = H(X) + H(Y|X)$$

Proof:

$$H(X,Y) = -\sum_{X,Y} P(x,y)\log_2 P(x,y) = -\sum_{X,Y} P(x,y)\log_2 P(x)P(y\,|\,x)$$

$$= -\sum_{X} P(x)\sum_{Y} P(y\,|\,x)\log_2 P(x) - \sum_{X,Y} P(x,y)\log_2 P(y\,|\,x) = H(X) + H(Y\,|\,X)$$

$$H(X\,|\,Y) = -\sum_{X,Y} P(x,y)\log_2 P(x\,|\,y)$$

Joint Entropy

Is the amount of information contained in both events X and Y

$$H(X,Y) = H(X) + H(Y|X)$$

Also $H(X,Y) = H(Y) + H(X|Y)$

- Intuition: first describe Y and then X given Y
- From this: $H(X) - H(X|Y) = H(Y) - H(Y|X)$

Joint Entropy

Example

$$H(X,Y) = H(X) + H(Y|X)$$

X	Y
Math	Yes
History	No
CS	Yes
Math	No
Math	No
CS	Yes
History	No
Math	Yes

v_j	$Prob(X=v_j)$	$H(Y \mid X = v_j)$
Math	0.5	1
History	0.25	0
CS	0.25	0

$H(X) = 1.5$

$H(Y) = 1$

$H(Y|X) = 0.5$

$$H(X,Y) = H(X) + H(Y|X) = 1.5 + 0.5 = 2$$

Joint Entropy

Comments

$$H(X,Y) = H(X) + H(Y|X)$$
$$= H(Y) + H(X|Y)$$

$$H(X,Y,Z) = H(X) + H(Y|X) + H(Z|XY)$$
$$\leq H(X) + H(Y) + H(Z)$$