Linear Discriminant Analysis

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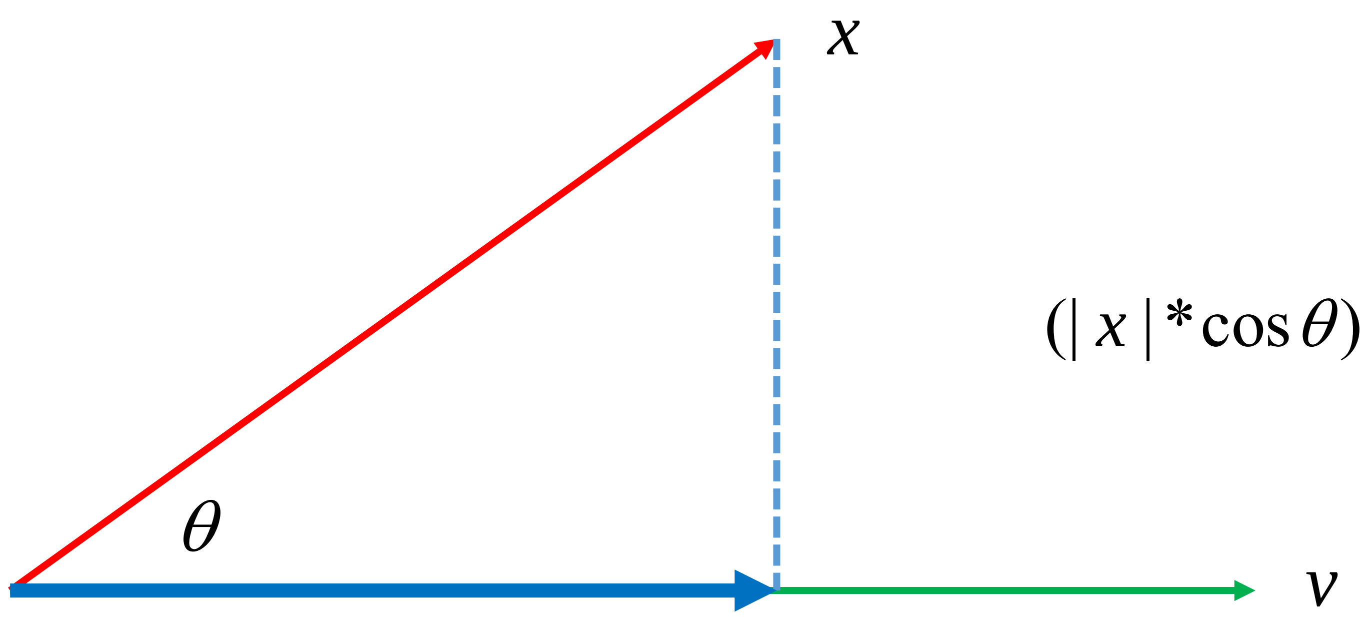
Vector Projection
--Review



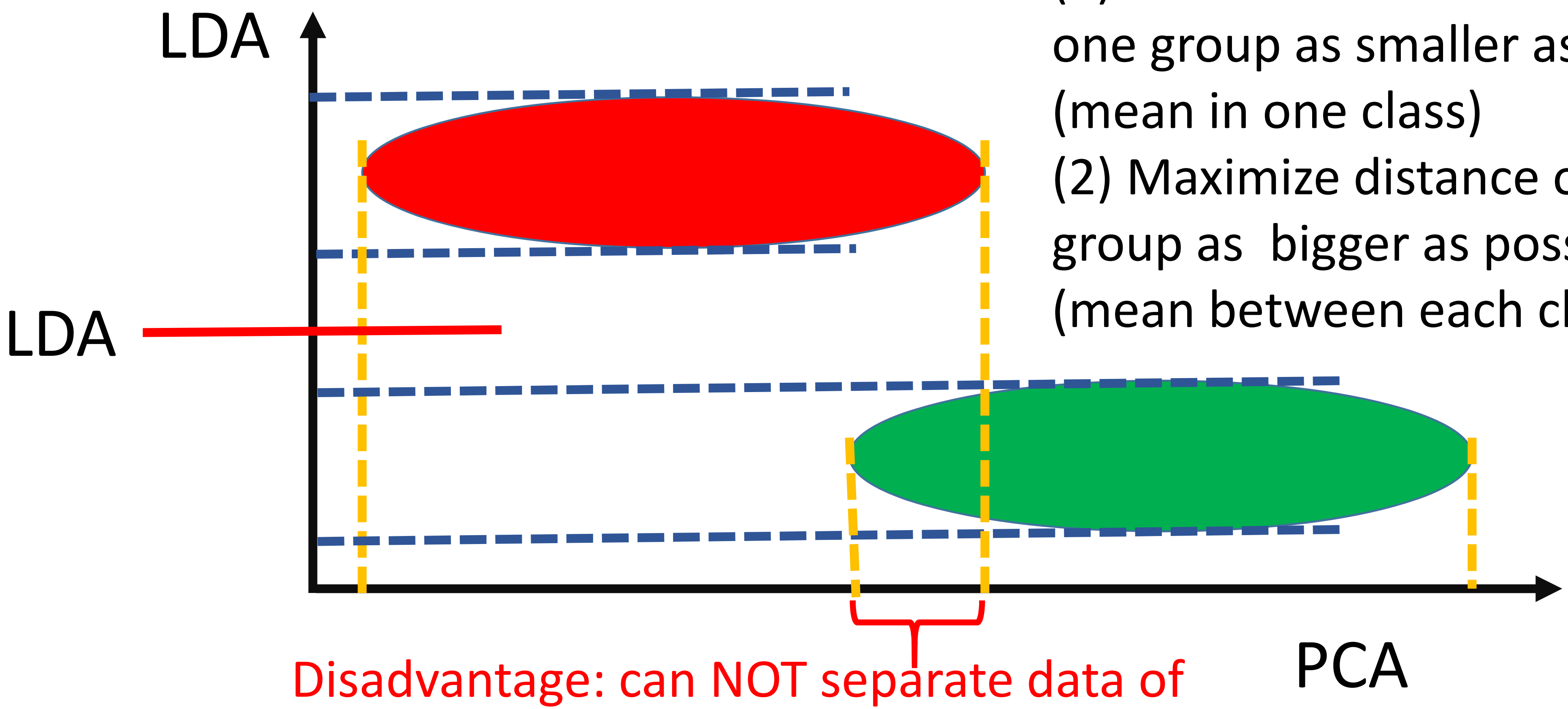
$$(|x|*\cos\theta)\frac{v}{|v|} = \ldots = \frac{<x, v>}{|v|^2} v$$

Linear Discriminate Analysis

Two Objectives of LDA:

(1) Minimize distance of data in one group as smaller as possible (mean in one class)

(2) Maximize distance of each group as bigger as possible (mean between each class)



Disadvantage: can NOT separate data of two group here!!!

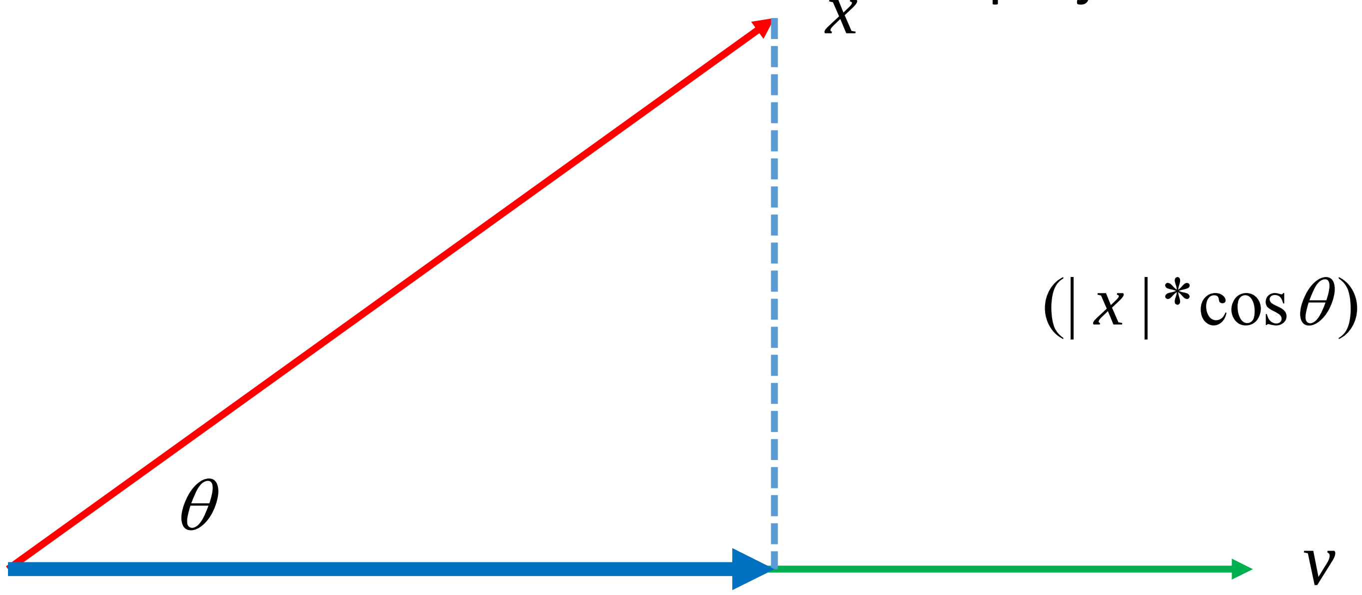
Linear Discriminate Analysis

- Notation
  - L : # of classes
  - N_i: # of samples in class i
  - N: # of all samples
  - $X^{(i)}_j$: the j-th sample in class i

- E.g.
  - i: $X^{(i)}_1$, $X^{(i)}_2$, …, $X^{(i)}_{N_i}$

Linear Discriminate Analysis

- Given data as $x_1^{(1)},\ldots,x_{N_1}^{(1)};x_1^{(2)},\ldots,x_{N_2}^{(2)};\ldots;x_1^{(L)},\ldots,x_{N_L}^{(L)}$
- Question:
- what is the coordinator of these data in projected direction v?



$$(|x|*\cos\theta)\frac{v}{|v|}=\ldots=\frac{<x,v>}{|v|^2}v$$

Linear Discriminate Analysis

- Given data as $x_1^{(1)},\ldots,x_{N_1}^{(1)};x_1^{(2)},\ldots,x_{N_2}^{(2)};\ldots;x_1^{(L)},\ldots,x_{N_L}^{(L)}$
- Question:
- what is the coordinator of these data in projected direction v?

$$v^T x_1^{(1)},\ldots,v^T x_{N_1}^{(1)};v^T x_1^{(2)},\ldots,v^T x_{N_2}^{(2)};\ldots;v^T x_1^{(L)},\ldots,v^T x_{N_L}^{(L)}$$

Linear Discriminate Analysis

- Two Objectives of LDA:
  - (1) Minimize distance of data as smaller as possible (mean in one class)
  - (2) Maximize distance of each group as bigger as possible (mean between each class)

- What is expression of the mean in one class?
- What is expression of the mean between each class?
- What is expression of total distance among mean and its data in one class?

derivation

Linear Discriminate Analysis --mean in one class

$$v^T x_1^{(1)}, \ldots, v^T x_{N_1}^{(1)}; v^T x_1^{(2)}, \ldots, v^T x_{N_2}^{(2)}; \ldots; v^T x_1^{(L)}, \ldots, v^T x_{N_L}^{(L)}$$

- What is expression of the mean in one class?

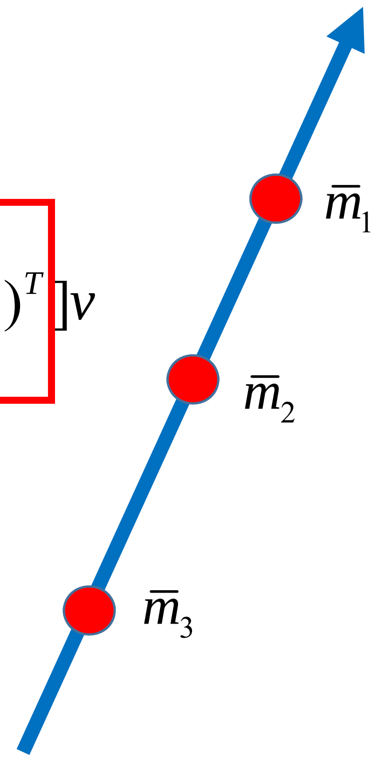
$$\bar{m}_i = \frac{1}{N_i}\sum_{j=1}^{N_i} v^T x_j^{(i)} = v^T \boxed{\frac{1}{N_i}\sum_{j=1}^{N_i} x_j^{(i)}} = v^T \boxed{m_i} \qquad m_i = \frac{1}{N_i}\sum_{j=1}^{N_i} x_j^{(i)}$$

derivation

Linear Discriminate Analysis --between class scatter matrix

$$v^T x_1^{(1)},\ldots,v^T x_{N_1}^{(1)};v^T x_1^{(2)},\ldots,v^T x_{N_2}^{(2)};\ldots;v^T x_1^{(L)},\ldots,v^T x_{N_L}^{(L)}$$

- What is expression of the mean between each class?



$$\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\frac{N_i}{N}\frac{N_j}{N}(\bar{m}_i-\bar{m}_j)^2=\ldots=v^T\left[\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\frac{N_i}{N}\frac{N_j}{N}(\bar{m}_i-\bar{m}_j)(\bar{m}_i-\bar{m}_j)^T\right]v$$

derivation

Linear Discriminate Analysis --between class scatter matrix

$$v^T x_1^{(1)},\dots,v^T x_{N_1}^{(1)};v^T x_1^{(2)},\dots,v^T x_{N_2}^{(2)};\dots;v^T x_1^{(L)},\dots,v^T x_{N_L}^{(L)}$$

- Let us simplify this equation

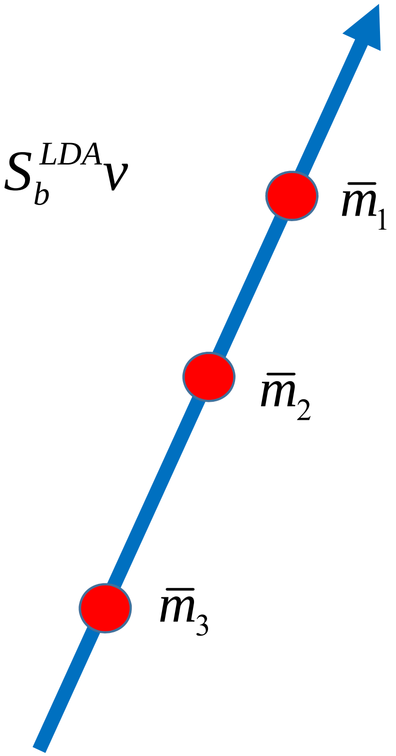
$$\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\frac{N_i}{N}\frac{N_j}{N}(\bar{m}_i-\bar{m}_j)^2=\dots=v^T\left[\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\frac{N_i}{N}\frac{N_j}{N}(m_i-m_j)(m_i-m_j)^T\right]v=v^T S_b^{LDA} v$$

derivation

$$S_b^{LDA}=\sum_{i=1}^{L}\frac{N_i}{N}(m_i-m_0)(m_i-m_0)^T$$

$$m_0=\sum_{i=1}^{L}\sum_{k=1}^{N_i}\frac{1}{N}x_k^{(i)}$$

M0 is the total mean

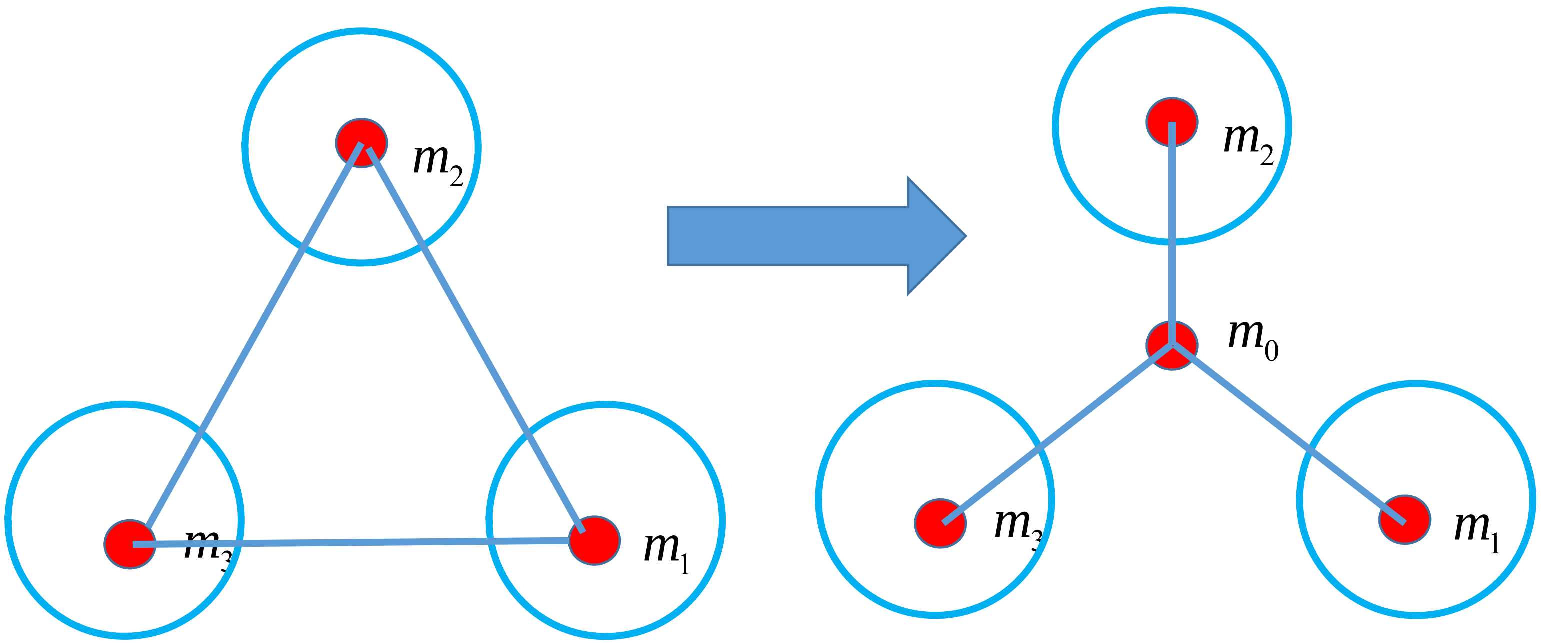


Linear Discriminate Analysis
--between class scatter matrix

- Visual explanation of S

$$S_b^{LDA} = \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \frac{N_i}{N} \frac{N_j}{N} (\overline{m}_i - \overline{m}_j)(\overline{m}_i - \overline{m}_j)^T = \sum_{i=1}^{L} \frac{N_i}{N} (m_i - m_0)(m_i - m_0)^T$$

$$m_0 = \sum_{i=1}^{L} \sum_{k=1}^{N_i} \frac{1}{N} x_k^{(i)}$$



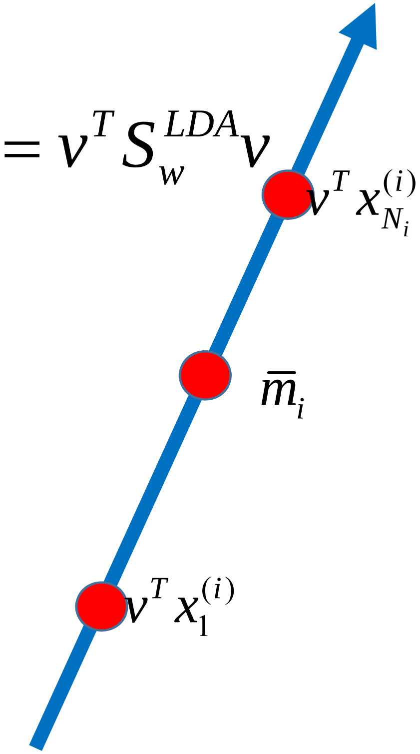
M0 is the total mean

Linear Discriminate Analysis --within class scatter matrix

- What is expression of total distance among mean and its data in one class?

$$\sum_{i=1}^{L}\sum_{j=1}^{N_i}\frac{1}{N}(v^T x_j^{(i)} - \bar{m}_i)^2 = \ldots = v^T\left[\sum_{i=1}^{L}\sum_{j=1}^{N_i}\frac{1}{N}(x_j^{(i)} - m_i)(x_j^{(i)} - m_i)^T\right]v = v^T S_w^{LDA} v$$

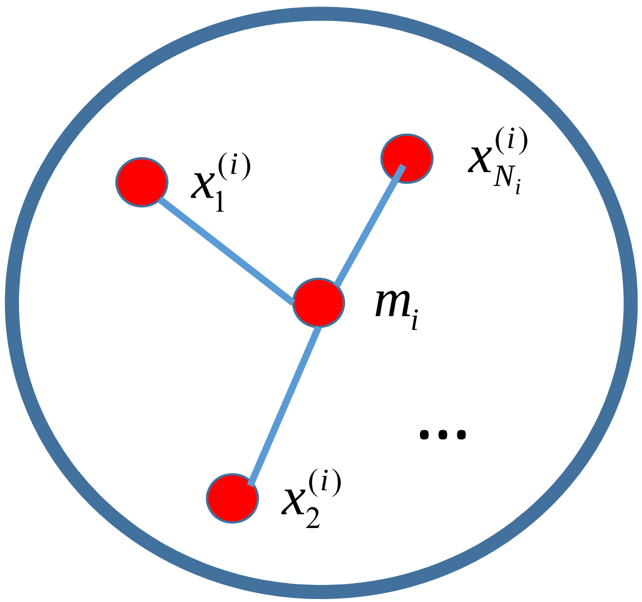
$$S_w^{LDA} = \sum_{i=1}^{L}\sum_{j=1}^{N_i}\frac{1}{N}(x_j^{(i)} - m_i)(x_j^{(i)} - m_i)^T$$



Linear Discriminate Analysis --within class scatter matrix

- Visual explanation of S

$$S_w^{LDA} = \sum_{i=1}^{L} \sum_{j=1}^{N_i} \frac{1}{N} (x_j^{(i)} - m_i)(x_j^{(i)} - m_i)^T$$



Linear Discriminate Analysis
--objective function

- What is the two objectives of Linear discriminate analysis?
  - (1) Minimize distance of data as smaller as possible (mean in one class)
  - (2) Maximize distance of each group as bigger as possible (mean between each class)

$$v^T S_b^{LDA} v = v^T [\sum_{i=1}^{L} \frac{N_i}{N} (m_i - m_0)(m_i - m_0)^T] v$$

$$v^T S_w^{LDA} v = v^T [\sum_{i=1}^{L} \sum_{j=1}^{N_i} \frac{1}{N} (x_j^{(i)} - m_i)(x_j^{(i)} - m_i)^T] v$$

- Question: Can we change it to an one objective function?

Linear Discriminate Analysis
--objective function

- The first principal vector can be defined by

$$v = \arg\max \frac{v^T S_b^{LDA} v}{v^T S_w^{LDA} v} = \underset{v^T S_w^{LDA} v = 1}{\arg\max}\, v^T S_b^{LDA} v$$

- Solution: Lagrangian

$$f(v,\lambda) = v^T S_b^{LDA} v - \lambda(v^T S_w^{LDA} v - 1)$$

- Please solve it.....

derivation

Linear Discriminate Analysis --objective function solution

$$f(v,\lambda) = v^T S_b^{LDA} v - \lambda(v^T S_w^{LDA} v - 1)$$

$$\frac{\partial f}{\partial v} = 2S_b^{LDA} v - 2\lambda S_w^{LDA} v = 0$$

$$\frac{\partial f}{\partial \lambda} = v^T S_w^{LDA} v - 1 = 0$$

$\Rightarrow$

$$(S_w^{LDA})^{-1} S_b^{LDA} v = \lambda v$$

$$v^T S_w^{LDA} v = 1$$

$$v = \arg\max \frac{v^T S_b^{LDA} v}{v^T S_w^{LDA} v} = \arg\max_{v^T S_w^{LDA} v = 1} v^T S_b^{LDA} v$$

$\Rightarrow$

$$v^T S_b^{LDA} v = \lambda v^T S_w^{LDA} v = \lambda$$

Linear Discriminate Analysis

- The first linear discriminant vector can be found by

$$v = \arg\max \frac{v^T S_b^{LDA} v}{v^T S_w^{LDA} v} = \underset{v^T S_w^{LDA} v = 1}{\arg\max}\, v^T S_b^{LDA} v$$

- This is to equivalent to find the largest eigenvalue of the following generalized eigenvalue problem:

$$S_b^{LDA} u = \lambda S_w^{LDA} u$$

$$v = \frac{1}{\sqrt{u^T S_w^{LDA} u}} u$$