Lecture 10

10.1 (Complex) Power Series (複数)冪級数

10.2 (Complex) Taylor Series (複数)テイラー級数

10.1 (Complex) Power Series

(複数)冪級数

Definition 1: Power Series (冪級数)

A **power series (centered at z_0)** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (6.1.11)$$

where **the coefficients (係数) a_n are complex constants.**

Theorem 1 Radius of Convergence (収束半径)

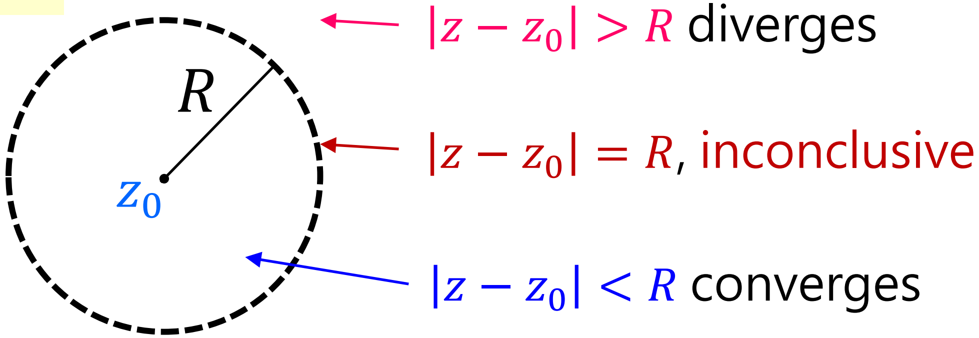
Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be **a power series (centered at z_0)**.

Then there is R, where $0 \le R \le +\infty$, we have

(i) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ **converges absolutely** if $|z - z_0| < R$.

(ii) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ **diverges** if $|z - z_0| > R$.

We call R **the radius of convergence of the series.**



EXAMPLE (例題) 6.1.5 **Circle of Convergence**

Evaluate the convergence condition of the power series $\sum_{n=1}^{\infty} \frac{z^{n+1}}{n}$.

Solution (解答):

By **the ratio test** (6.1.9) $\lim_{n\to\infty} \left| \frac{\frac{z^{(n+1)+1}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{z^{n+2}}{z^{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |z| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} |z| = |z|$

Thus the series **converges absolutely** for $L = |z| < 1$.

The circle of convergence is $|z| = 1$ and then the **radius of convergence** is $R = 1$.

Note that on the circle of convergence $|z| = 1$, the series does not converge absolutely

because $\sum_{n=1}^{\infty} \frac{1}{n}$ is the **well-known divergent harmonic series**.

This *does not say* that the series diverges on the circle of convergence.

In fact, at $z = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is **the convergent alternating harmonic series.**

It can be shown that **the series converges at all points on the circle** $|z| = 1$ **except at** $z = 1$.

Notice the radius of convergence for Ratio Test

It should be clear from **Theorem 6.4** in Lecture 9 and **Example 6.1.5** that **for a power series** $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, **the limit** (6.1.9) depends only on the coefficients a_n. Thus, if

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, **the radius of convergence** is $R = \frac{1}{L}$. (6.1.12)

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, then the **radius of convergence** $R = \infty$. (6.1.13)

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, **the test is inconclusive** $R = 0$. (6.1.14)

EXAMPLE (例題) 6.1.6 **Radius of Convergence by Ratio Test**

For the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}(z-1-i)^n$, find its radius of convergence and the condition that it converges absolutely.

$(z-(1+i))^n$

Solution (解答):

Identify that $a_n = \frac{(-1)^{n+1}}{n!}$ then

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\frac{(-1)^{n+2}}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n!}{n!\,(n+1)}\frac{(-1)^{n+2}}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Hence by (6.1.13) the **radius of convergence** R is ∞;
the **power series** with center $z_0 = 1+i$ **converges absolutely** for all z, that is, for $|z-(1+i)| < \infty$.

Additional EXAMPLE (例題) 1 **Radius of Convergence by Ratio Test**

Find the radius of convergence of the following power series:

(1) $\sum_{n=0}^{\infty} 2^n z^n$ (2) $\sum_{n=0}^{\infty} \frac{n}{6^n} z^n$ (3) $\sum_{n=0}^{\infty} n^2 z^n$

Solution (解答):

By (6.1.12),

(1) $R = \frac{1}{L} = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{2^{n+1}}\right| = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$

(2) $R = \frac{1}{L} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{6^n}\frac{6^{n+1}}{n+1}\right| = \lim_{n\to\infty}\left|\frac{n}{6^n}\frac{6^n\cdot 6}{n+1}\right| = \lim_{n\to\infty}\left|\frac{6n}{n+1}\right| = \lim_{n\to\infty}\frac{6}{1+\frac{1}{n}} = 6$

(3) $R = \frac{1}{L} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$

10.1 (Complex) Power Series (複数)冪級数

Notice the radius of convergence for Root Test

Similar conclusions can be made for **the root test (6.1.10)** by using

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \qquad (6.1.15)$$

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0$, **the radius of convergence** is $R = \frac{1}{L}$.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$, then the **radius of convergence** $R = \infty$.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, **the test is inconclusive** $R = 0$.

EXAMPLE (例題) 6.1.7 Radius of Convergence by Root Test

For the power series $\sum_{n=1}^{\infty}\left(\frac{6n+1}{2n+5}\right)^n (z-2i)^n$, find its radius of convergence and the condition that it converges absolutely.

Solution (解答):

Identify that $a_n = \left(\frac{6n+1}{2n+5}\right)^n$ then

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{6n+1}{2n+5} = \lim_{n\to\infty}\frac{6+\frac{1}{n}}{2+\frac{5}{n}} = 3$$

By previous slides, we conclude that the **radius of convergence of the series** is $R = \frac{1}{3}$.

The circle of convergence is $|z-2i| = \frac{1}{3}$;

the power series **converges absolutely** for $|z-2i| < \frac{1}{3}$.

10.2 (Complex) Taylor Series

(複数)テイラー級数

10.2 (Complex) Taylor Series (複数)テイラー級数

James Gregory
ジェームス・グレゴリー
(1638–1675)

Brook Taylor
ブルック・テイラー
(1685-1731)

Colin Maclaurin
コリン・マクローリン
(1698–1746)

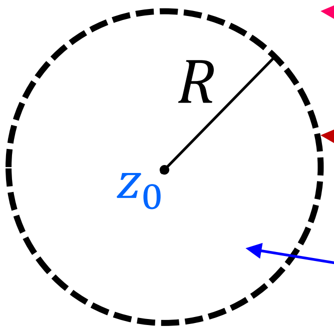
Background:
The **Taylor Series** subject was formulated by the Scottish mathematician **James Gregory** and formally introduced by the English mathematician **Brook Taylor** in 1715. If the Taylor series is centered at zero, then that series is also called a Maclaurin series, after the Scottish mathematician **Colin Maclaurin**, who made extensive use of this special case of Taylor series in the 18th century.

10.2 (Complex) Taylor Series (複数)テイラー級数

Notice: A power series *defines* or *represents* a function f.

Theorem 6.6 Continuity

A power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function f within its circle of convergence $|z - z_0| = R$, namely, $|z - z_0| < R$.



$|z - z_0| > R$ diverges

$|z - z_0| = R$, inconclusive

$|z - z_0| < R$ converges

Differentiation (微分) and Integration (積分) of Power Series

Theorem 6.7 Term-by-Term (項別に) Differentiation

A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated term-by-term within its circle of convergence $|z-z_0| = R$, namely, $|z-z_0| < R$.

Differentiating a power series term-by-term gives,

$$\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1}$$

Note that the summation index in the last series starts with $n = 1$ because the term differentiation corresponding to $n = 0$ is zero.

Differentiation (微分) and Integration (積分) of Power Series

Theorem 6.8 Term-by-Term Integration

A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ can be integrated term-by-term within its circle of convergence $|z - z_0| = R$ (namely, $|z - z_0| < R$), for every contour C lying entirely within the circle of convergence.

This theorem gives that

$$\int_C \sum_{n=0}^{\infty} a_n (z - z_0)^n \, dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n \, dz$$

whenever C lies in the interior of $|z - z_0| = R$.

10.2 (Complex) Taylor Series (複数)テイラー級数

Taylor Series

Suppose a power series represents a function f within $|z - z_0| = R$, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \tag{6.2.1}$$

It follows from Theorem 6.7 that the derivatives of f are the series

$$f'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1} = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots \tag{6.2.2}$$

$$f''(z) = \sum_{n=2}^{\infty} a_n n(n-1)(z - z_0)^{n-2} = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3(z - z_0) + \cdots \tag{6.2.3}$$

$$f'''(z) = \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(z - z_0)^{n-3} = 3 \cdot 2 \cdot 1 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(z - z_0) + \cdots \tag{6.2.4}$$

and so on.

10.2 (Complex) Taylor Series (複数)テイラー級数

Taylor Series

A power series represents an analytic function within its circle of convergence.

There is a relationship between the coefficients a_n in (6.2.1) and the **derivatives of f**. Evaluating (6.2.1), (6.2.2), (6.2.3), and (6.2.4) at $z = z_0$ we have

$$f(z_0) = a_0, \qquad f'(z_0) = 1!\,a_1, \qquad f''(z_0) = 2!\,a_2, \qquad f'''(z_0) = 3!\,a_3,$$

In general, $f^{(n)}(z_0) = n!\,a_n,$ or

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \; n \geq 0 \tag{6.2.5}$$

Taylor Series

When $n = 0$ in (6.2.5), we interpret **the zero-order derivative** as $f(z_0)$ and $0! = 1$, so that the formula gives $a_0 = f(z_0)$.

Substituting (代入する) (6.2.5) into (6.2.1), we have

Definition 2: Taylor series (テイラー級数)

The **Taylor series** for f **centered at** z_0 is of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{6.2.6}$$

Maclaurin Series (マクローリン級数)

Definition 3: Maclaurin series (マクローリン級数)

The Maclaurin series is a special Taylor series with center $z_0 = 0$, i.e.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \qquad (6.2.7)$$

10.2 (Complex) Taylor Series (複数)テイラー級数

Question

If we are **given a function** f that **is** **analytic in some domain** D, can we **represent it by a power series of the form** (6.2.6) or (6.2.7)?

Check the answer in the following Theorem 6.9.

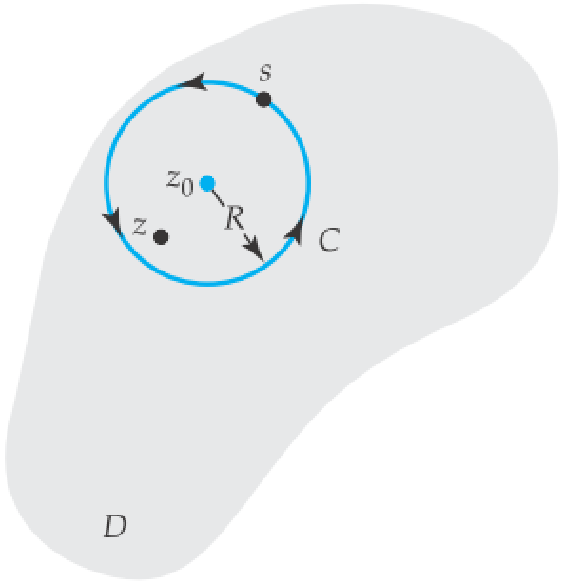
10.2 (Complex) Taylor Series (複数)テイラー級数

Theorem 6.9 Taylor's Theorem

Let f **be** **analytic** **within a domain** D and let z_0 be a point in D. Then f has the series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \qquad (6.2.8)$$

which is valid for the largest circle C with center at z_0 and radius R that lies entirely within D.



Some Important Maclaurin Series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (6.2.12)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (6.2.13)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad (6.2.14)$$

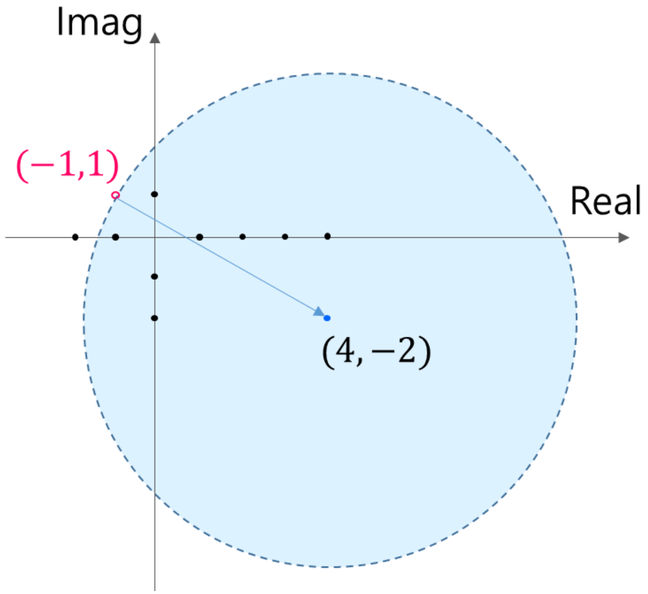
EXAMPLE (例題) 6.2.1 Radius of Convergence

Suppose the function $f(z)=\frac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0=4-2i$. What is its radius of convergence R?

Solution (解答):

Observe that the function is analytic at every point except at $z=-1+i$, which is an isolated singularity of f. The distance from $z=-1+i$ to $z_0=4-2i$ is

$$|z-z_0|=\sqrt{(-1-4)^2+\big(1-(-2)\big)^2}=\sqrt{34}$$



This last number **is the radius of convergence** R for the Taylor series centered at $4-2i$.

EXAMPLE (例題) 6.2.2 Maclaurin Series

Find the Maclaurin expansion of $f(z) = \frac{1}{(1-z)^2}$.

Solution (解答):

We could begin **by computing the coefficients** using (6.2.8). However, recall from (6.1.6) in Lecture 9 that for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{6.1.6}$$

If we differentiate both sides with respect to z, then

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}1 + \frac{d}{dz}z + \frac{d}{dz}z^2 + \frac{d}{dz}z^3 + \cdots$$

or

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{n=1}^{\infty} nz^{n-1}$$

Since we are using Theorem 6.7, the radius of convergence of the last power series is the same as the original series, $R = 1$.

EXAMPLE (例題) 6.2.3 Taylor Series

Expand $f(z)=\frac{1}{1-z}$ in a Taylor series with center $z_0=2i$.

Solution (解答):

We use the **geometric series** (6.1.6) in Lecture 9.
By adding and subtracting $2i$ in the denominator of $1/(1-z)$, we can write

$$\frac{1}{1-z}=\frac{1}{1-z+2i-2i}=\frac{1}{1-2i-(z-2i)}=\frac{1}{1-2i}\frac{1}{1-\frac{z-2i}{1-2i}}$$

We now write $\frac{1}{1-\frac{z-2i}{1-2i}}$ as a **power series** by using (6.1.6) with that z replaced by $\frac{z-2i}{1-2i}$

$$\frac{1}{1-z}=\frac{1}{1-2i}\left[1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\cdots\right] \qquad (6.2.17)$$

Because the distance from the center $z_0=2i$ to the nearest singularity $z=1$ is $\sqrt{5}$,
we conclude that the circle of convergence for (6.2.17) is $|z_0-2i|=\sqrt{5}$.
This can be verified by the **ratio test** of the Lecture 9.

Review for Lecture 10

- (Complex) Power Series
- Radius of Convergence
- (Complex) Taylor Series
- (Complex) Maclaurin Series

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia