Lecture 13

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

13.2 Some Consequences of the Residue Theorem

13.1 Residues (留数) &

Residue Theorem (留数定理)

Part 2

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

Theorem 6.14 Residue at a Simple Pole

If f has a simple pole at $z = z_0$, then

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \qquad (6.5.1)$$

Theorem 6.15 Residue at a Pole of Order n

If f has a pole of order n at $z = z_0$, then

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \qquad (6.5.2)$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

When f **is not a rational function**, calculating **residues by means of** (6.5.1) or (6.5.2) in Lecture 12 can sometimes be tedious.

It is possible to devise **alternative residue formulas**.

In particular, suppose a function f can be written **as a quotient**

$f(z) = \frac{g(z)}{h(z)}$, where g and h **are analytic at** $z = z_0$.

If $g(z_0) \neq 0$ and if the function h has **a zero of order** 1 **at** z_0, then f has **a simple pole at** $z = z_0$ and

$$a_{-1} = \text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{6.5.4}$$

13.1 Residues (留数) & Residue Theorem (留数定理) Part 2

To derive this result we shall use **the definition of a zero of order 1**, **the definition of a derivative**, and then (6.5.1).

First, since the function h **has a zero of order 1 at** z_0, we must have $h(z_0) = 0$ and $h'(z_0) \neq 0$.

Second, by definition of the **derivative** given in (3.1.12) of Lecture 3,

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0}$$

We then combine the preceding two facts in the following manner in (6.5.1):

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Recall in Lecture 5

Roots of a Complex Number

Consider to find z in $z^k = w$

where z and w are complex numbers,

k is real, i.e. NOT **a complex number**.

then

$$z = \sqrt[k]{|w|}\left[\cos\left(\frac{\arg(w) + 2n\pi}{k}\right) + i\sin\left(\frac{\arg(w) + 2n\pi}{k}\right)\right] \quad (1.4.4)$$

where $n = 0, 1, 2, \dots, k-1$

EXAMPLE (例題) 6.5.3 Using (6.5.4) to Compute Residues

The polynomial $z^4 + 1$ can be factored as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where z_1, z_2, z_3, and z_4 are the four distinct roots of the equation $z^4 + 1 = 0$ (or equivalently, the four fourth roots of -1). It follows from Theorem 6.13 that the function

$$f(z) = \frac{1}{z^4 + 1}$$

has **four simple poles**. By using (6.5.4), find its **residues**.

Solution (解答):

From (1.4.4) of Lecture 5, for $z^4 + 1 = 0$, we have $z^4 = -1$.

Thus for $n = 0, 1, 2, 3$, we obtain $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{5\pi i/4}$, and $z_4 = e^{7\pi i/4}$.

To compute the **residues**, we use (6.5.4) of this Lecture by identifying $g(z) = 1$, $h(z) = z^4 + 1$,

along with **Euler's formula** (1.6.6) $e^{\theta} = \cos\theta + i\sin\theta$

Solution (解答)(cont.):

$$a_{-1} = \mathrm{Res}(f(z), z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_2) = \frac{g(z_2)}{h'(z_2)} = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_3) = \frac{g(z_3)}{h'(z_3)} = \frac{1}{4z_3^3} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_4) = \frac{g(z_4)}{h'(z_4)} = \frac{1}{4z_4^3} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

Theorem 6.16 Cauchy's Residue Theorem

Let D be **a simply connected domain** and C **a simple closed contour lying entirely within** D. If a function f **is analytic on and within** C, **except at a finite number of isolated singular points** $z_1, z_2, \ldots, z_n$ **within** C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \qquad (6.5.5)$$

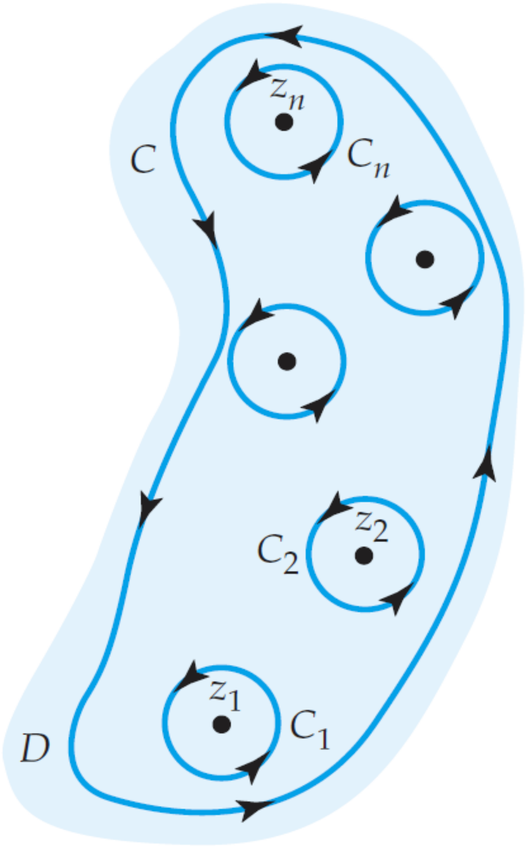


Figure 6.10 n **singular points within contour** C

Proof of Theorem 6.16

Suppose $C_1, C_2, \ldots, C_n$ are circles centered at $z_1, z_2, \ldots, z_n$, respectively.

Suppose further that each circle C_k has a radius r_k small enough so that $C_1, C_2, \ldots, C_n$ are **mutually disjoint** and are **interior to** the **simple closed curve** C.

See Figure 6.10. Now given

$$\oint_C f(z)dz = 2\pi i a_{-1} \tag{6.3.20}$$

we saw that $\oint_{C_k} f(z)dz = 2\pi i \operatorname{Res}(f(z), z_k)$, and so by Theorem 5.5 of Lecture 7 we have

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

EXAMPLE (例題) 6.5.4 **Evaluation by the Residue Theorem**

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$, where

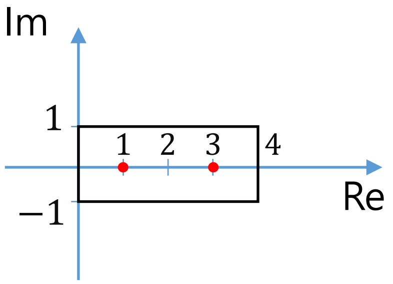
(a) the contour C is the rectangle defined by $x = 0, x = 4, y = -1, y = 1$,

(b) and the contour C is the circle $|z| = 2$.

Solution (解答):

(a) Since both $z = 1$ and $z = 3$ are **poles** within the rectangle we have from (6.5.5) that

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i [\text{Res}(f(z), 1) + \text{Res}(f(z), 3)]$$

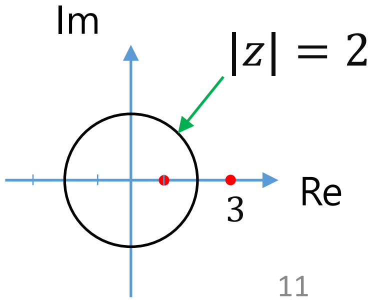


We found **these residues** in Example 6.5.2 of Lecture 12. Therefore,

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i \left[\left(-\frac{1}{4}\right) + \frac{1}{4}\right] = 0$$

(b) Since only the **pole** $z = 1$ lies within the circle $|z| = 2$, we have from (6.5.5)

$$\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i \, \text{Res}(f(z), 1) = 2\pi i \left(-\frac{1}{4}\right) = -\frac{\pi}{2} i$$



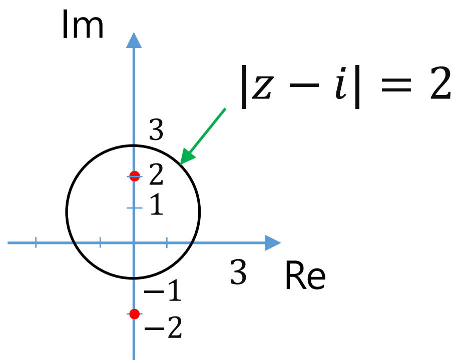
EXAMPLE (例題) 6.5.5 **Evaluation by the Residue Theorem**

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$, where the contour C is the circle $|z-i|=2$.

Solution (解答):

By factoring the **denominator** as $z^2+4=(z-2i)(z+2i)$ we see that the integrand has **simple poles** at $-2i$ and $2i$. Because only $2i$ lies within the contour C, it follows from (6.5.5) that

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\ \mathrm{Res}(f(z), 2i)$$



But
$$\mathrm{Res}(f(z),2i) = \lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)} = \frac{3+2i}{2i}$$

Hence,
$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\left(\frac{3+2i}{2i}\right) = \pi(3+2i)$$

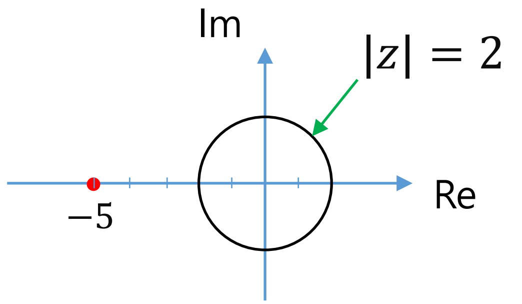
EXAMPLE (例題) 6.5.6 **Evaluation by the Residue Theorem**

Evaluate $\oint_C \frac{e^z}{z^4+5z^3}dz$, where the contour C is the circle $|z| = 2$.

Solution (解答):

Writing the denominator as $z^4 + 5z^3 = z^3(z + 5)$ reveals that the integrand $f(z)$ has **a pole of order** 3 at $z = 0$ and **a simple pole** at $z = -5$.

But only **the pole** $z = 0$ lies within **the circle** $|z| = 2$ and so from (6.5.5) and (6.5.2) we have,



$$\oint_C \frac{e^z}{z^4 + 5z^3}dz = 2\pi i \, \mathrm{Res}(f(z), 0) = 2\pi i \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} z^3 \frac{e^z}{z^3(z+5)}$$

$$= \pi i \lim_{z \to 0} \frac{(z^2 + 8z + 17)e^z}{(z+5)^3} = \frac{17\pi}{125} i$$

EXAMPLE (例題) 6.5.7 Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z\, dz$, where the contour C is the circle $|z| = 2$.

Solution (解答):

The integrand $f(z) = \tan z = \sin z / \cos z$ has **simple poles** at the points where $\cos z = 0$. We saw in Lecture 5 that the only **zeros** $\cos z$ are the real numbers $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \dots$. Since only $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are within the circle $|z| = 2$, we have

$$\oint_C \tan z\, dz = 2\pi i \left[\operatorname{Res}\left(f(z), -\frac{\pi}{2}\right) + \operatorname{Res}\left(f(z), \frac{\pi}{2}\right)\right]$$

With the identifications $g(z) = \sin z$, $h(z) = \cos z$, and $h'(z) = -\sin z$, we see from (6.5.4) that

$$\operatorname{Res}\left(f(z), -\frac{\pi}{2}\right) = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1 \quad \text{and} \quad \operatorname{Res}\left(f(z), \frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$$

Therefore, $\oint_C \tan z\, dz = 2\pi i\,[-1 - 1] = -4\pi i$

EXAMPLE (例題) 6.5.8 Evaluation by the Residue Theorem

Evaluate $\oint_C e^{\frac{3}{z}}\,dz$, where the contour C is the circle $|z| = 1$.

Solution (解答):

As we have seen, $z = 0$ is an **essential singularity** of the **integrand** $f(z) = e^{\frac{3}{z}}$ and so neither formulas (6.5.1) and (6.5.2) are applicable to find the **residue** of f at that point.

We saw in Example 6.5.1(b) of Lecture 12 that the **Laurent series** of f at $z = 0$ gives $\mathrm{Res}(f(z), 0) = 3$.

Hence from (6.5.5) we have

$$\oint_C e^{\frac{3}{z}}\,dz = 2\pi i\,\mathrm{Res}(f(z), 0) = 2\pi i \cdot 3 = 6\pi i$$

*13.2 Some Consequences of the Residue Theorem

Notice: In all lectures notes, the contents marked with * are not in the scope of the final examination.

*13.2 Some Consequences of the Residue Theorem

Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta\,, \sin\theta)\,d\theta$ (6.6.1)

The basic idea here is to convert **a real trigonometric integral of form** (6.6.1) into **a complex integral**, where the contour C is the unit circle $|z| = 1$ **centered at the origin**.

To do this we begin with (2.2.10) of Lecture 7 to **parametrize this contour by** $z = e^{i\theta},\ 0 \le \theta \le 2\pi$. We can then write

$$dz = ie^{i\theta}d\theta \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

*13.2 Some Consequences of the Residue Theorem

The last two expressions follow from (4.3.2) and (4.3.3) of Lecture 5.

Since $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ and $z^{-1} = 1/z = e^{-i\theta}$, **these three quantities** are equivalent to

$$d\theta = \frac{dz}{iz} \qquad \cos\theta = \frac{z + z^{-1}}{2} \qquad \sin\theta = \frac{z - z^{-1}}{2i} \tag{6.6.4}$$

The conversion of the **integral** in $\int_0^{2\pi} F(\cos\theta\,, \sin\theta)\, d\theta$ into **a contour integral** is **accomplished by replacing**, in turn, $d\theta$, $\cos\theta$, and $\sin\theta$ by the expressions in (6.6.4):

$$\oint_C F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz}$$

where C is **the unit circle** $|z| = 1$.

***EXAMPLE (例題) 6.6.1 A Real Trigonometric Integral**

Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$

Solution (解答):

When we use the substitutions given in (6.6.4), **the given trigonometric integral** becomes **the contour integral**

$$\oint_C \frac{1}{\left(2+\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz} = \oint_C \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}$$

Carrying out **the algebraic simplification** of **the integrand** then yields

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}dz$$

From **the quadratic formula** we can **factor the polynomial** z^2+4z+1 as $z^2+4z+1=(z-z_1)(z-z_2)$, where $z_1=-2-\sqrt{3}$ and $z_2=-2+\sqrt{3}$.

Solution (解答)(cont.):

Thus, **the integrand** can be written

$$\frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_1)^2(z-z_2)^2}$$

Because only z_2 is inside **the unit circle** C, we have

$$\oint_C \frac{z}{(z^2+4z+1)^2} dz = 2\pi i \, \mathrm{Res}(f(z), z_2)$$

To calculate **the residue**, we first note that z_2 is **a pole of order** 2 and so we use (6.5.2) of this Lecture:

$$\mathrm{Res}(f(z), z_0) = \lim_{z\to z_2} (z-z_0)^2 f(z) = \lim_{z\to z_2} \frac{d}{dz}\frac{1}{(z-z_0)^2}$$

$$= \lim_{z\to z_2} \frac{-z-z_1}{(z-z_0)^3} = \frac{1}{6\sqrt{3}}$$

Solution (解答)(cont.):

Hence,

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\,dz = \frac{4}{i}\cdot 2\pi i \ \mathrm{Res}(f(z), z_1) = \frac{4}{i}\cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$$

and, finally,

$$\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta = \frac{4\pi}{3\sqrt{3}}$$

Review for Lecture 13

- Residues (留数)
- Residue Theorem (留数定理)

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia