Lecture 3

3.1 Differentiability (微分可能性) & Analyticity (解析性)

3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

3.1 Differentiability (微分可能性) & Analyticity (解析性)

3.1 Differentiability (微分可能性) & Analyticity (解析性)

Derivative (導関数)

Definition (定義) 3.1 Derivative (導関数) of Complex Function (複素関数)

Suppose (仮定する) the complex function f is defined in a neighborhood (近傍) of a point z_0. The derivative (導関数) of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (3.1.1)$$

when this limit exists.

- If the limit in (3.1.1) exists, then the function f is said to be differentiable (微分可能) at z_0.
- Besides $f'(z_0)$, we have two other symbols denoting (表示する) the derivative of $w = f(z)$, which are w' and $\dfrac{dw}{dz}$.

EXAMPLE (例題) 3.1.1

Use Definition 3.1 to find the derivative of $f(z) = z^2 - 5z$.

Solution (解答):

We replace z_0 in (3.1.1) by the symbol z for any point. First, compute the complex function

$$f(z+\Delta z) = (z+\Delta z)^2 - 5(z+\Delta z)$$
$$= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z.$$

Second,

$$f(z+\Delta z) - f(z) = z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z - (z^2 - 5z)$$
$$= 2z\Delta z + (\Delta z)^2 - 5\Delta z.$$

Finally, (3.1.1) gives

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} (2z + \Delta z - 5)$$

⇨ The limit is $f'(z) = 2z - 5$

3.1 Differentiability (微分可能性) & Analyticity (解析性)

Rules of Differentiation (微分法則)

If f and g are **differentiable** at a point z, and c is a **complex constant**, then (3.1.1) can be used to show:

Constant Rules (定数の法則): $\frac{d}{dz}c = 0$ and $\frac{d}{dz}cf(z) = cf'(z)$ (3.1.2)

Sum Rule (和の法則): $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$ (3.1.3)

Product Rule (積の法則): $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$ (3.1.4)

Quotient Rule (商の法則): $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$ (3.1.5)

Chain Rule (連鎖律): $\frac{d}{dz}f(g(z)) = f'\big(g(z)\big)g'(z)$ (3.1.6)

Power Rule (冪乗の法則): $\frac{d}{dz}z^n = nz^{n-1}$, where n is an integer. (3.1.7)

Combine (3.1.7) with (3.1.6), $\frac{d}{dz}[g(z)]^n = n[g(z)]^{n-1}g'(z)$, n is an integer. (3.1.8)

EXAMPLE (例題) 3.1.2 Using the Rules of Differentiation.
Differentiate:

(a) $f(z) = 3z^4 - 5z^3 + 2z$ (b) $f(z) = \frac{z^2}{4z+1}$ (c) $f(z) = (iz^2 + 3z)^5$

Solution (解答):

(a) Using the power rule (3.1.7), the sum rule (3.1.3), along with (3.1.2), we obtain

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 \cdot 1 = 12z^3 - 15z^2 + 2$$

(b) From the quotient rule (3.1.5),

$$f'(z) = \frac{2z \cdot (4z+1) - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$$

(c) In the power rule for functions (3.1.8) we identify $n = 5$, $g(z) = iz^2 + 3z$, and $g'(z) = 2iz + 3$, so that

$$f'(z) = 5(iz^2 + 3z)^4(2iz + 3)$$

EXAMPLE (例題) 3.1.3 A Function That is Nowhere Differentiable. Show that the function $f(z) = x + 4yi$ is **not differentiable** at any point z.

Solution (解答):

Let z be any point in the complex plane. With $\Delta z = \Delta x + i\Delta y$,

$$f(z+\Delta z) - f(z) = (x+\Delta x) + 4(y+\Delta y)i - (x+4yi) = \Delta x + 4\Delta yi$$

and so

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\Delta x + 4\Delta yi}{\Delta x + \Delta yi} \qquad (3.1.9)$$

Now, as shown in Figure 3.1(a), if we let $\Delta z \to 0$ along a line parallel to the x-axis, then $\Delta y = 0$ and $\Delta z = \Delta x$ and

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\Delta x}{\Delta x} = 1 \qquad (3.1.10)$$

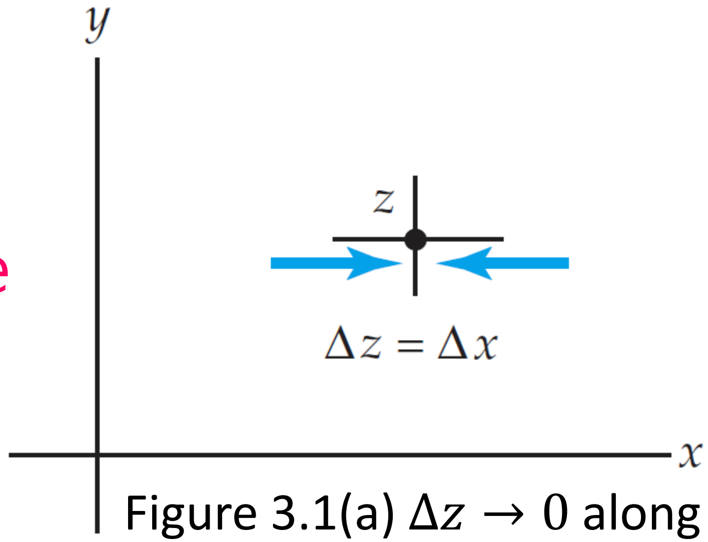

Figure 3.1(a) $\Delta z \to 0$ along a line parallel to x-axis

3.1 Differentiability (微分可能性) & Analyticity (解析性)

On the other hand, if we let $\Delta z \to 0$ along a line parallel to the y-axis as shown in Figure 3.1(b), then $\Delta x = 0$ and $\Delta z = i\Delta y$ so that

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{4\Delta y i}{\Delta y i} = 4 \qquad (3.1.11)$$

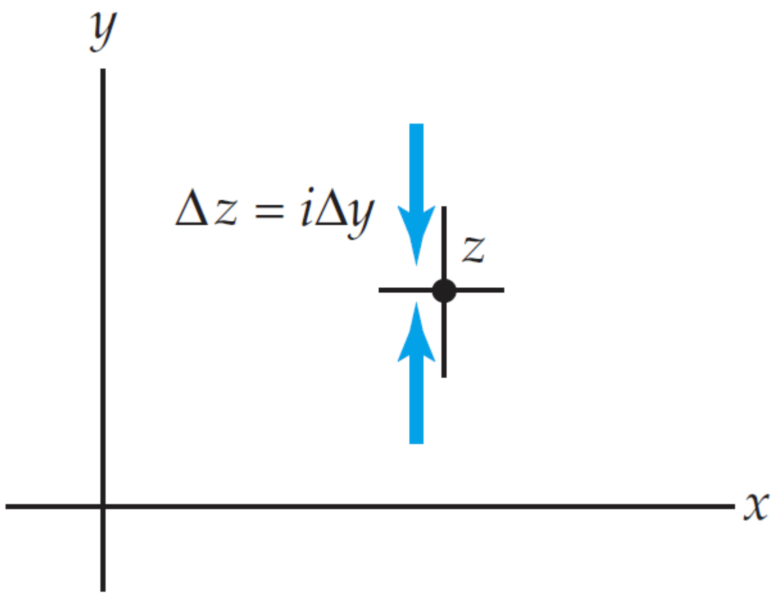

Figure 3.1(b) $\Delta z \to 0$ along a line parallel to y-axis

In view of the obvious fact that the values in (3.1.10) and (3.1.11) are different, we conclude that $f(z) = x + 4yi$ is nowhere differentiable; that is, f is not differentiable at any point z.

Analytic Functions (解析関数)

Definition (定義) 3.2 Analyticity (解析性) at a Point

A **complex function** (複素関数) $w = f(z)$ is said to be **analytic (解析的)** at a point z_0 if f is **differentiable (微分可能)** at z_0 and **at every point in some neighborhood of** z_0.

- A function f **is analytic in a domain** D if it is analytic at every point in D.

  Notice: A function f that is **analytic** throughout a domain D is also called **holomorphic function (正則関数)** or **regular function**.

- Analyticity is a neighborhood property that is defined over an open set (開集合) (i.e. not only for a single point).

3.1 Differentiability (微分可能性) & Analyticity (解析性)

If the functions f and g are **analytic** in a domain D, then

Analyticity of Sum (和), Product (積), and Quotient (商)

The sum $f(z)+g(z)$, difference (差) $f(z)-g(z)$, and product $f(z)g(z)$ are analytic. The quotient $\frac{f(z)}{g(z)}$ is analytic if $g(z)\neq 0$ in D.

3.1 Differentiability (微分可能性) & Analyticity (解析性)

Entire Functions (整函数)

A function that is analytic at every point z in the complex plane is said to be an entire function (整函数).

Theorem 3.1 Polynomial and Rational Functions

(i) A **polynomial function (多項式関数)** $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a **nonnegative integer**, is an **entire function (整函数)**.

(ii) A **rational function (有理関数)** $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are **polynomial functions (多項式関数)**, is **analytic** in **any domain** D that **contains** no point z_0 **for which** $q(z_0) = 0$.

Singular Points (特異点)

In general, a point z at which a complex function $w = f(z)$ **fails (失敗する) to be analytic** is called **a singular point (特異点)** of f.

For example, since the rational function $f(z) = \frac{4z}{z^2-2z+2}$ is discontinuous at $1+i$ and $1-i$ because $z^2 - 2z + 2 = 0$, f fails to be analytic at these points.

Thus by (ii) of Theorem 3.1, f is **not analytic** in any domain containing one or both of these points.

These two points are **singular points** of f.

Theorem 3.2 Differentiability (微分可能性) Implies (伴う) Continuity (連続性)

If f is **differentiable (微分可能)** at a point z_0 in a domain D, then f is **continuous (連続)** at z_0.

An Alternative (代わりの) Definition of the Derivative (導関数) $f'(z)$

We know

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \text{from (3.1.1)}$$

Since $\Delta z = z - z_0$, then $z = z_0 + \Delta z$, and so (3.1.1) can be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (3.1.12)$$

Theorem 3.3 L'Hôpital's Rule (ロピタルの定理)

Suppose f and g are functions that are **analytic** at a point z_0 and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{3.1.13}$$

EXAMPLE (例題) 3.1.4 Using L'Hôpital's Rule (ロピタルの定理)

Compute $\lim_{z\to 2+i} \frac{z^2-4z+5}{z^3-z-10i}$.

Solution (解答):

We know $z_0 = 2+i$.

If we identify $f(z) = z^2 - 4z + 5$ and $g(z) = z^3 - z - 10i$, you should verify that

$$f(z_0) = f(2+i) = (2+i)^2 - 4(2+i) + 5 = 4 + 4i + i^2 - 8 - 4i + 5 = 0$$

$$g(z_0) = g(2+i) = (2+i)^3 - (2+i) - 10i = (4+4i+i^2)(2+i) - 2 - i - 10i$$
$$= (3+4i)(2+i) - 2 - 11i = 6 + 8i + 3i + 4i^2 - 2 - 11i = 0$$

The given limit has the indeterminate form $\frac{0}{0}$.

Now since f and g are polynomial functions, both functions are necessarily analytic at $z_0 = 2+i$.

$$f'(z) = \frac{d(z^2-4z+5)}{dz} = 2z - 4, \quad g'(z) = \frac{d(z^3-z-10i)}{dz} = 3z^2 - 1,$$

then $f'(2+i) = 2(2+i) - 4 = 2i, \quad g'(2+i) = 3(2+i)^2 - 1 = 8 + 12i$

we see that (3.1.13) gives $\lim_{z\to 2+i} \frac{z^2-4z+5}{z^3-z-10i} = \frac{f'(2+i)}{g'(2+i)} = \frac{2i}{8+12i} = \frac{2i(8-12i)}{(8+12i)(8-12i)} = \frac{3}{26} + \frac{1}{13}i$

Exercise (練習): Using L'Hôpital's Rule (ロピタルの定理)

Compute $\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}$.

Solution (解答):

We know $z_0 = 1 + \sqrt{3}i$.

If we identify $f(z) = z^2 - 2z + 4$ and $g(z) = z - 1 - \sqrt{3}i$, verify that

$f(z_0) = f(1 + \sqrt{3}i) = 0$

$g(z_0) = g(1 + \sqrt{3}i) = 0$

The given limit has the indeterminate form $\frac{0}{0}$.

Since f and g are polynomial functions, both functions are necessarily analytic at $z_0 = 1 + \sqrt{3}i$.

$$f'(z) = \frac{d(z^2 - 2z + 4)}{dz} = 2z - 2, \quad g'(z) = \frac{d\left(z - 1 - \sqrt{3}i\right)}{dz} = 1,$$

we see that (3.1.13) gives $\lim_{z \to 1+\sqrt{3}i} \frac{z^2 - 2z + 4}{z - 1 - \sqrt{3}i} = \frac{f'(1 + \sqrt{3}i)}{g'(1 + \sqrt{3}i)} = \frac{2(1 + \sqrt{3}i) - 2}{1} = 2\sqrt{3}i$

3.2 Cauchy-Riemann Equations

(コーシー・リーマンの方程式)

A Necessary Condition (必要条件) for Analyticity (解析性)

Theorem 3.4 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order (一階) partial derivatives of $u(x, y)$ and $v(x, y)$ exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (3.2.1)$$

Proof: P152 of Textbook

Alternatively, $u_x = v_y$ and $u_y = -v_x$

EXAMPLE (例題) 3.2.1 Verifying Cauchy-Riemann Equations for the polynomial function $f(z) = z^2 + z$

Solution (解答):

The polynomial function $f(z) = z^2 + z$ is analytic for all z and can be written as

$f(z) = (x + iy)^2 + (x + iy) = x^2 - y^2 + x + i(2xy + y)$.

Thus, $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$.

For any point (x, y) in the complex plane, we can see that the **Cauchy-Riemann equations** are satisfied:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x - 0 + 1 \\ \frac{\partial v}{\partial y} &= 2x + 1 \end{aligned}\right\} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \left.\begin{aligned} \frac{\partial u}{\partial y} &= 0 - 2y + 0 \\ \frac{\partial v}{\partial x} &= 2y + 0 \end{aligned}\right\} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

Criterion (基準) for Non-Analyticity

If the Cauchy-Riemann equations are NOT satisfied at every point z in a domain D,

then the function $f(z) = u(x, y) + iv(x, y)$ CANNOT be analytic in D.

EXAMPLE (例題) 3.2.2 Using the Cauchy-Riemann Equations

Show that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

Solution (解答):

We identify $u(x, y) = 2x^2 + y$ and $v(x, y) = y^2 - x$. From

$$\frac{\partial u}{\partial x} = 4x + 0, \quad \frac{\partial v}{\partial y} = 2y - 0$$

If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ then we have

$$y = 2x$$

and

$$\frac{\partial u}{\partial y} = 0 + 1, \quad \frac{\partial v}{\partial x} = 0 - 1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we see that $\partial u/\partial y = -\partial v/\partial x$ but that the equality $\partial u/\partial x = \partial v/\partial y$ is satisfied only on the line $y = 2x$.

However, by Definition 3.2, for any point z on the line, there is no neighborhood or open disk about z in which f is differentiable at every point. We conclude that f is not analytic at any point.

A **Sufficient Condition (十分条件)** for **Analyticity** (解析性)

Theorem 3.5 Criterion (基準) for Analyticity

Suppose the **real functions** $u(x, y)$ and $v(x, y)$ are **continuous** and have **continuous first-order partial derivatives** $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$, in a domain D.

If u and v satisfy the Cauchy-Riemann equations (3.2.1) at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

EXAMPLE (例題) 3.2.3 Using Theorem 3.5 to evaluate the analyticity of the function $f(z)=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$

Solution (解答):

For the function $f(z)=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$, the real functions $u(x,y)=\frac{x}{x^2+y^2}$ and $v(x,y)=-\frac{y}{x^2+y^2}$ are continuous except at the point where $x^2+y^2=0$, that is, at $z=0+i0=0$.

Moreover, we can verify that the first four first-order partial derivatives

$$\frac{\partial u}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y}=-\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2}, \quad \text{and} \quad \frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

are continuous except at $z=0$.

3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

Solution (解答)(cont.):

Finally, we see from

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

that the Cauchy-Riemann equations are satisfied except at $z = 0$.

Thus we conclude from Theorem 3.5 that f is analytic in any domain D that does not contain the point $z = 0$.

We call this $z = 0$ a singular point (特異点).

3.2 Cauchy-Riemann Equations (コーシー・リーマンの方程式)

Sufficient Conditions (十分条件) for Differentiability (微分可能性)

If the real functions $u(x, y)$ and $v(x, y)$ are continuous and also have continuous first-order partial derivatives in some neighborhood of a point z, and **if** u and v **satisfy the Cauchy-Riemann equations** (3.2.1) **at** z, then **the complex function** $f(z) = u(x, y) + iv(x, y)$ is **differentiable at** z and $f'(z)$ is given by (3.2.9).

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{3.2.9}$$

Namely, we can apply (3.2.9) to find the derivative $f'(z)$.

Statement: Analyticity implies Differentiability but not conversely.

Analyticity $\Rightarrow$ / $\not\Leftarrow$ Differentiability

EXAMPLE (例題) 3.2.4 A Function Differentiable on a Line

Solution (解答):

In Example 3.2.2 we saw that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ was nowhere analytic, but yet the Cauchy-Riemann equations were satisfied on the line $y = 2x$.

Since the functions $u(x,y) = 2x^2 + y$, $\partial u/\partial x = 4x$, $\partial u/\partial y = 1$, $v(x,y) = y^2 - x$, $\partial v/\partial x = -1$ and $\partial v/\partial y = 2y$ are **continuous** at every point, it follows that f is **differentiable** on the line $y = 2x$.

Moreover, from (3.2.9) we see that the derivative of f at points on this line is given by $f'(z) = 4x - i = 2y - i$.

Review for Lecture 3

- Differentiability (微分可能性)
- Analyticity (解析性)
- Holomorphic function (正則関数)
- Singular Point (特異点)
- L'Hôpital's Rule (ロピタルの定理)
- Cauchy-Riemann Equations (コーシー・リーマンの方程式)
- Criterion (基準) for Analyticity

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia