# Lecture 4

**4.1** Harmonic Functions (調和関数)

**4.2** (Complex) Elementary Functions (複素)初等関数 Part 1:

**4.2.1** (Complex) Exponential Functions (複素)指数関数

**4.2.2** (Complex) Logarithmic Functions (複素)対数関数

# 4.1 Harmonic Functions (調和関数)

# 4.1 Harmonic Functions (調和関数)

## Laplace's Equation (ラプラス方程式)

The **second-order partial differential equation (二階偏微分方程式)**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (3.3.1)$$

is called **Laplace's equation (ラプラス方程式)** in two independent variables $x$ and $y$.

( The sum of the two second partial derivatives in (3.3.1) is denoted by $\nabla^2 \phi$ and is called the **Laplacian** of $\phi(x, y)$. **Laplace's equation** is then abbreviated as $\nabla^2 \phi = 0$. )

**Definition 3.3 Harmonic Functions (調和関数)**

A real-valued function $\phi$ of two real variables $x$ and $y$ that has **continuous (連続) first and second-order partial derivatives (一階と二階偏微分) in a domain $D$** and satisfies **Laplace's equation** is said to be **harmonic in $D$**.

## Harmonic Functions (調和関数)

### Theorem 3.7 Analyticity (解析性) and Harmonic Functions (調和関数)

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain $D$.

then $u(x,y)$ is harmonic in $D$

and $v(x,y)$ is harmonic in $D$.

Proof: The Page 160 of Textbook

## Harmonic Functions (調和関数)

**EXAMPLE (例題) 3.3.1 Harmonic Functions**
Show that the real and imaginary parts of function $f(z) = z^2$, where $z = x + iy$, are harmonic in $\mathbf{C}$.

**Solution (解答):**

The function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **entire (i.e. 整函数)**.

Then the function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is **analytic at every point** $z$ in the complex plane.

According to Theorem 3.7,
The functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily **harmonic** in the complex plane, i.e. in $\mathbf{C}$.

# 4.1 Harmonic Functions (調和関数)

## Harmonic Conjugate Functions (共役調和関数)

Now suppose $u(x, y)$ is a given real function that **is harmonic** in $D$;

↓

find another real harmonic function $v(x, y)$ so that $u$ and $v$ satisfy the **Cauchy-Riemann equations** throughout the domain $D$;

↓

then this function $v(x, y)$ is called a **harmonic conjugate function (共役調和関数)** of $u(x, y)$.

↓

By combining the functions as $u(x, y) + iv(x, y)$, we obtain a function $f(z)$ that is **analytic** in $D$.

## Harmonic Conjugate Functions (共役調和関数)

**EXAMPLE (例題) 3.3.2 Harmonic Conjugate Function**

(a) Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is **harmonic in the entire complex plane.**

(b) Find **the harmonic conjugate function** of $u(x, y)$.

**Solution (解答):**

(**a**) From the partial derivatives

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy - 5 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

we see that $u$ **satisfies Laplace's equation**: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$

Therefore, according to the Definition 3.3, $u(x, y)$ is **harmonic** in **C**.

**Solution (解答)(cont.):**

(b) Since the **harmonic conjugate function** $v$ must **satisfy the Cauchy-Riemann equations** $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ then we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy - 5) = 6xy + 5 \qquad (3.3.3)$$

**Partial integration (積分)** of $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ **with respect to** $y$ gives

$$v(x, y) = 3x^2y - y^3 + h(x).$$

From this $v(x, y)$, we compute the **partial derivative with respect to** $x$ as

$$\frac{\partial v}{\partial x} = 6xy + h'(x)$$

Compare this $\frac{\partial v}{\partial x}$ with the second equation in (3.3.3), we can obtain $h'(x) = 5$, and so $h(x) = 5x + c$, where $c$ is a real constant.

Therefore, the **harmonic conjugate function** of $u(x, y)$ is $v(x, y) = 3x^2y - y^3 + 5x + c$.

# 4.2 (Complex) Elementary Functions (複素)初等関数 Part 1 :

## 4.2.1 (Complex) Exponential Functions (複素)指数関数

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

Suppose we know the fact that $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, where $\alpha$ and $\beta$ are complex numbers.

**Definition 4.1 Complex Exponential Function (複素指数函数)**

The function $e^z$ (where $z = x + iy$) defined by

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

By Euler's Formula

i.e. $$e^z = e^x \cos y + ie^x \sin y \qquad (4.1.1)$$

is called the **complex exponential function**.

**Notice**: The function defined by (4.1.1) agrees with the real exponential function, i.e.

if $z$ is real number, then $z = x + 0i$, and Definition 4.1 gives:

$$e^{x+i0} = e^x(\cos 0 + i\sin 0) = e^x(1 + i0) = e^x$$

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

## Theorem 4.1 Analyticity (解析性) of $e^z$

The **exponential function** $e^z$ is **entire** and **its derivative is given by:**

$$\frac{d}{dz}e^z = e^z \tag{4.1.3}$$

Proof: The Page 177 of Textbook

**EXAMPLE (例題) 4.1.1 Derivatives of Exponential Functions**

Find the derivative of the following functions:

(a) $iz^4(z^2 - e^z)$ and (b) $e^{z^2-(1+i)z+3}$

**Solution (解答):**

(a) Using Equation (4.1.3) and the product rule (積の法則) (3.1.4) in Lecture 3:

Product Rule (積の法則): $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$ (3.1.4)

$$\frac{d}{dz}[iz^4(z^2 - e^z)] = i4z^3(z^2 - e^z) + iz^4(2z - e^z)$$
$$= i6z^5 - iz^4e^z - i4z^3e^z$$

(b) Using Equation (4.1.3) and the chain rule (連鎖律) (3.1.6) in Lecture 3:

Chain Rule (連鎖律): $\frac{d}{dz}f(g(z)) = f'\big(g(z)\big)g'(z)$ (3.1.6)

$$\frac{d}{dz}\left[e^{z^2-(1+i)z+3}\right] = e^{z^2-(1+i)z+3} \cdot \big(2z - (1+i)\big) = e^{z^2-(1+i)z+3} \cdot (2z - 1 - i)$$

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

## Theorem 4.2 Properties (性質) of $e^z$

If $z_1$ and $z_2$ are complex numbers, then

(i) $e^0 = 1$

(ii) $e^{z_1}e^{z_2} = e^{z_1+z_2}$

(iii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

(iv) $(e^{z_1})^n = e^{nz_1}, n = 0, \pm 1, \pm 2, \ldots$

(v) $|e^z| = e^{\mathrm{Re}(z)}$, $\arg(e^z) = \mathrm{Im}(z) + 2n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$

(vi) $\overline{e^z} = e^{\bar{z}}$

(vii) $e^z \neq 0$, for all $z \in \mathbf{C}$

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

## Modulus (複素数の絶対値) and Argument (偏角)

We have the complex number $w = f(z) = e^z$ in polar form $re^{i\theta}$:

$$w = e^x \cos y + ie^x \sin y = e^x(\cos y + i\sin y) = r(\cos\theta + i\sin\theta)$$

then we see that the modulus $r = e^x$ and the argument $\theta = y + 2n\pi$, for $n = 0, \pm1, \pm2, \ldots$

**Modulus** $|e^z| = r = e^x = e^{\mathrm{Re}(z)}$ (4.1.4)

**Argument** $\arg(e^z) = \theta = y + 2n\pi = \mathrm{Im}(z) + 2n\pi$ for $n = 0, \pm1, \pm2, \ldots$ (4.1.5)

## Conjugate (複素共役)

Because $\cos(-y) = \cos y$ $\sin(-y) = -\sin y$

$$\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\bar{z}} \quad (4.1.6)$$

## Nonzero (非ゼロ)

From (4.1.4), we know $|e^z| > 0$ because $e^x > 0$ for all $x \in \mathbf{R}$. Then it implies $e^z \neq 0$, for all $z \in \mathbf{C}$.

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

## Periodicity (周期性)

$$e^{z+2\pi i} = e^z$$

The complex exponential function $e^z$ is **periodic** with a **pure imaginary period (純虚数周期)** $2\pi i$.

This is because, by (4.1.1) and Theorem 4.2(ii), we have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i\sin 2\pi) = e^z \cdot 1 = e^z$$

Notice that $e^{z+4\pi i} = e^{(z+2\pi i)+2\pi i} = e^{z+2\pi i} = e^z$

By repeating this process we find that

$e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \dots$

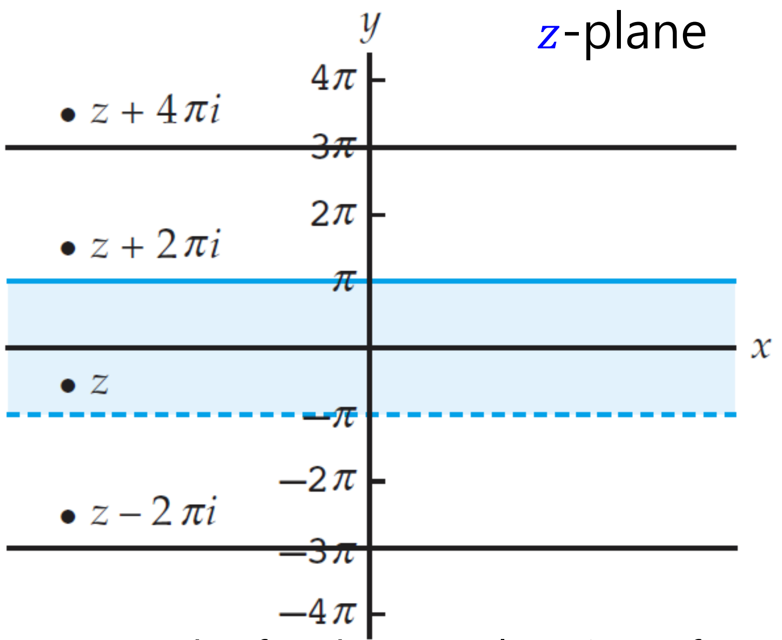


Figure 4.1 The fundamental region of $e^z$

# 4.2.1 (Complex) Exponential Functions (複素)指数関数

## **Fundamental Region** of the complex exponential function

- Because $e^{z+2n\pi i} = e^z$ for $n = 0, \pm 1, \pm 2, \ldots$

  thus there are many points in the $z$-plane, for example, $z - 2\pi i, z + 4\pi i, z + 6\pi i$, ...will correspond to the same single point $w = e^z$ in the $w$-plane,

  i.e. the complex exponential function $w = f(z) = e^z$ is not one-to-one (一対一) mapping from $z$-plane to $w$-plane.

- **We divide the complex plane into horizontal strips.**

- The **infinite horizontal (水平な) strip** defined by:

$$-\infty < x < \infty, -\pi < y \leq \pi$$

  is called the **fundamental region (基本領域)** of the complex exponential function $e^z$.

  Notice $f(z) = e^z, f(z + 2\pi i) = e^{z+2\pi i}, f(z - 2\pi i) = e^{z-2\pi i}$ and so on are the same.

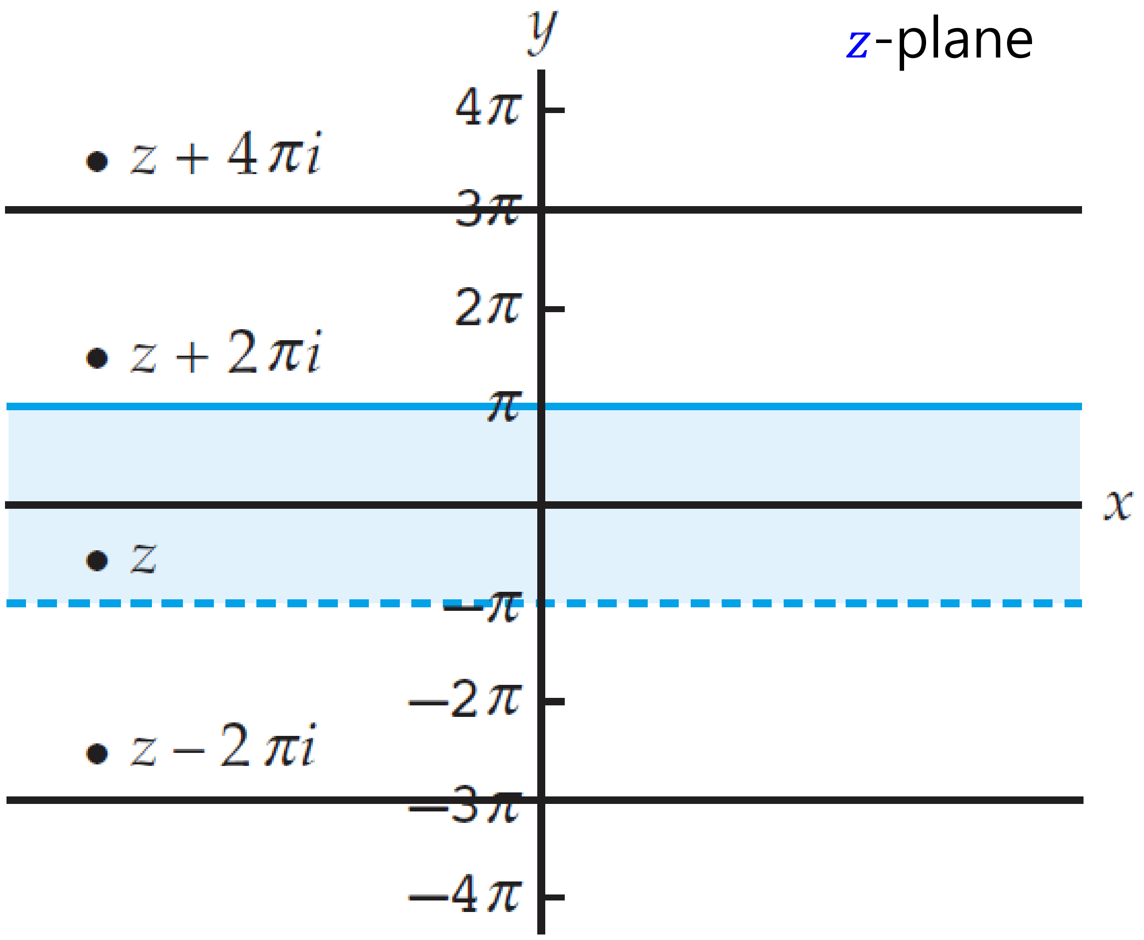
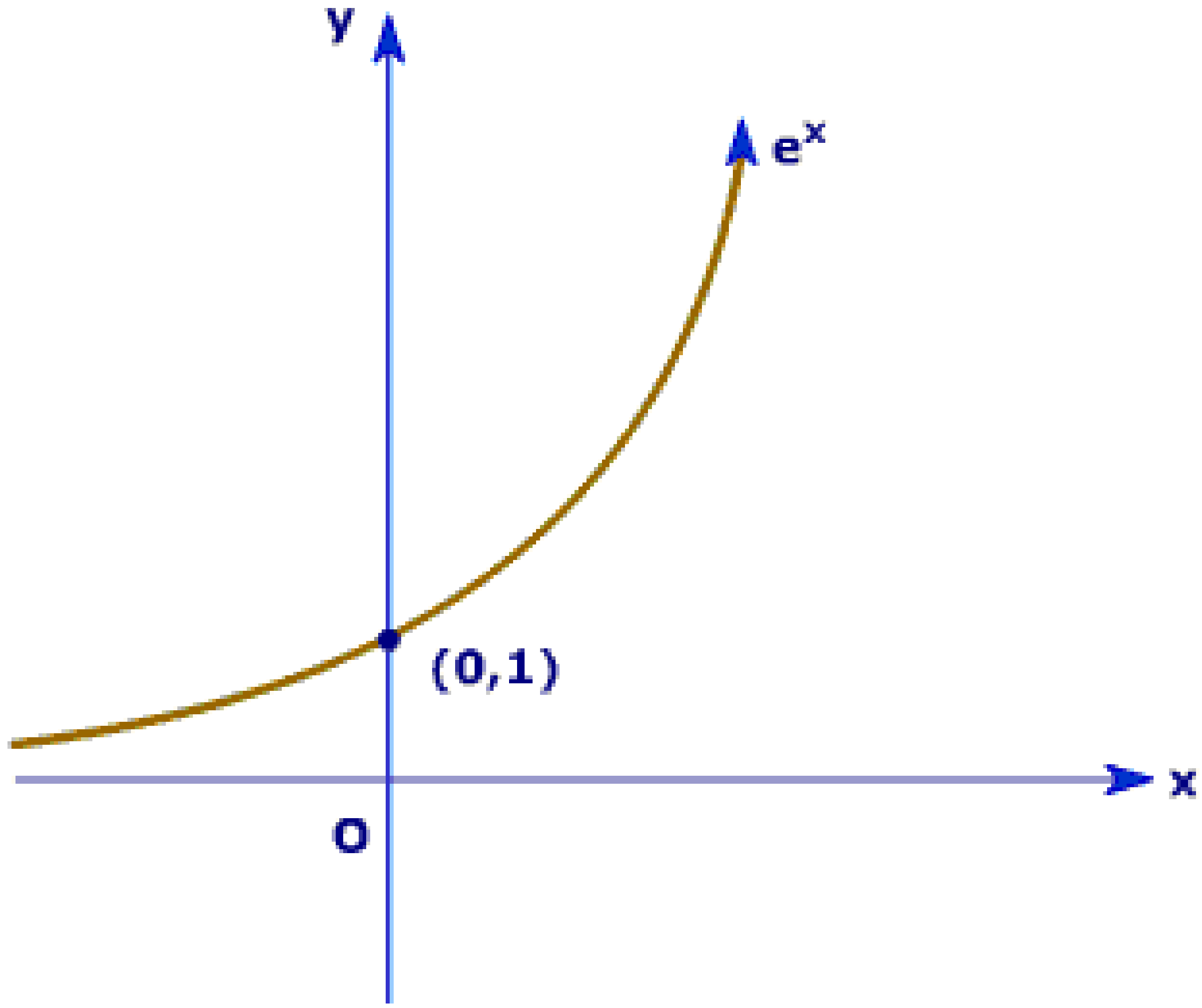



Figure 4.1 The fundamental region of $e^z$

## 4.2 (Complex) Elementary Functions (複素)初等関数 Part 1 :

### 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数




However, if $e^x = -2$,

then **solution**?

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

In **real** domain, the **natural logarithm function** $\ln x$ is often defined as an **inverse function (逆関数)** of **the real exponential function** $e^x$.

From now on, we can use the alternative notation $\log_e x$, where $x \in \mathbf{R}$ to represent the real exponential function $e^x$.

- The **real exponential function** is **one-to-one (一対一) on its domain $\mathbf{R}$**,
- **But the complex exponential function** $e^z$ **is NOT a one-to-one function on its domain $\mathbf{C}$**, because there are **infinitely (無限に) many arguments (偏角)** of $z$.

Note: **One-to-one (一対一) function** is a function that maps distinct elements of its domain to distinct elements of its range, i.e. $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

## 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

If $e^w = z$, then $w = \log_e |z| + i \arg(z)$ (4.1.10)

Because of the Periodicity (周期性), there are infinitely (無限に) many arguments (偏角) of $z$, thus **(4.1.10) gives infinitely many solutions** $w$ to the equation $e^w = z$.

The set of values given by (4.1.10) defines a multiple-valued function as:

**Definition 4.2 Complex Logarithm Function (複素対数関数)**

The multiple-valued function $\ln z$ (where $z = x + iy$) defined by

$$\ln z = \log_e |z| + i \arg(z) \qquad (4.1.11)$$

is called the **complex logarithm**.

**Notice**: We use the lowercase (小文字) letter for symbol $\ln z$.

**EXAMPLE (例題) 4.1.3 Solving Exponential Equations**

Find all complex solutions to each of the following equations.

(a) $e^w = i$ (b) $e^w = 1 + i$ (c) $e^w = -2$

**Solution (解答):**

For each equation $e^w = z$, the set of solutions is given by $w = \ln z$ where $\ln z$ is found using Definition 4.2.

(a) For $e^w = z = i$, we have $|z| = |i| = 1$ and $\arg(i) = \arctan(\frac{\mathrm{Im}(i)}{\mathrm{Re}(i)}) = \frac{\pi}{2} + 2n\pi$.

Thus, from (4.1.11) we obtain:

Because $\lim_{a\to\infty} \arctan(a) = \frac{\pi}{2}$

$$w = \ln i = \log_e |i| + i \arg(i)$$

$$= \log_e 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = 0 + i\left(\frac{\pi}{2} + 2n\pi\right) = \frac{(4n+1)\pi}{2} i \quad n = 0, \pm 1, \pm 2, \ldots$$

Therefore, each of the values $w = \cdots, -\frac{3\pi}{2}i, \frac{\pi}{2}i, \frac{5\pi}{2}i, \ldots$ satisfies the equation $e^w = i$ .

**Solution (解答)(cont.):**

(b) For $z = 1 + i$, we have $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{4} + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(1 + i) = \log_e |1 + i| + i \arg(1 + i)$$
$$= \log_e \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$$
$$= \frac{1}{2}\log_e 2 + \frac{(8n + 1)\pi}{4} i \qquad n = 0, \pm 1, \pm 2, \dots$$

Each value of $w$ is a solution to $e^w = 1 + i$.

(c) For $z = -2$, we have $|-2| = 2$ and $\arg(-2) = \arctan(\frac{\mathrm{Im}(-2)}{\mathrm{Re}(-2)}) = \pi + 2n\pi$.

Thus, from (4.1.11) we obtain:

$$w = \ln(-2) = \log_e |-2| + i \arg(-2)$$
$$= \log_e 2 + i(\pi + 2n\pi)$$
$$= \log_e 2 + (2n + 1)\pi i \qquad n = 0, \pm 1, \pm 2, \dots$$

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

**Theorem 4.3 Algebraic Properties of $\ln z$**

If $z_1$ and $z_2$ are nonzero complex numbers and $n$ is an integer, then

(i) $$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

(ii) $$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

(iii) $$\ln z_1^n = n \ln z_1$$

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

**Definition 4.3 Principal Value (主値) of the Complex Logarithm**

The complex function $\mathrm{Ln}\, z$ (where $z = x + iy$) defined by

$$\mathrm{Ln}\, z = \log_e |z| + i\, \mathrm{Arg}(z),\ -\pi < \arg(z) \leq \pi \qquad (4.1.14 \text{ and } 4.1.15)$$

is called **the principal value (主値) of the complex logarithm**.

**Notice**: We use the uppercase (大文字) letter for $\mathrm{Ln}\, z$ here!

**EXAMPLE (例題) 4.1.4 Principal Value of the Complex Logarithm**

Compute the principal value of the complex logarithm $\mathrm{Ln}\, z$ for

(a) $z = i$ (b) $z = 1 + i$ (c) $z = -2$

**Solution (解答):**

(a) For $e^w = z = i$, we have $|i| = 1$ and $\mathrm{Arg}(i) = \arctan(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}) = \frac{\pi}{2}$ ~~$+ 2n\pi$~~

Thus, from (4.1.14) we obtain:

$$
\begin{aligned}
w = \mathrm{Ln}\, i &= \log_e |i| + i\,\mathrm{Arg}(i) \\
&= \log_e 1 + \frac{\pi}{2} i \\
&= 0 + \frac{\pi}{2} i \\
&= \frac{\pi}{2} i
\end{aligned}
$$

~~$n = 0, \pm 1, \pm 2, \ldots$~~

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

**Solution (解答)(cont.):**

(b) For $e^w = z = 1 + i$, we have $|1 + i| = \sqrt{2}$ and $\text{Arg}(1 + i) = \arctan(\frac{\text{Im}(1+i)}{\text{Re}(1+i)}) = \frac{\pi}{4}$.

Thus, from (4.1.14) we obtain:

$$w = \text{Ln}(1 + i) = \log_e |1 + i| + i\,\text{Arg}(1 + i)$$

$$= \log_e \sqrt{2} + \frac{\pi}{4} i = \frac{1}{2} \log_e 2 + \frac{\pi}{4} i \approx \underline{0.3466 + 0.7854i}$$

Not necessary for the assignment report.

(c) For $e^w = z = -2$, we have $|-2| = 2$ and $\text{Arg}(-2) = \arctan(\frac{\text{Im}(-2)}{\text{Re}(-2)}) = \pi$.

Thus, from (4.1.14) we obtain:

$$w = \text{Ln}(-2) = \log_e |-2| + i\,\text{Arg}(-2)$$

$$= \log_e 2 + \pi i \approx \underline{0.6931 + 3.1416i}$$

Not necessary for the assignment report.

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

## $\mathrm{Ln}\, z$ as an Inverse Function (逆関数) of $e^z$

Follows from (4.1.10) that

$$e^{\mathrm{Ln}\, z} = z \text{ for all } z \neq 0. \tag{4.1.16}$$

If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty, -\pi < y \leq \pi$,

then $f$ is **one-to-one (一対一)** and the **inverse function (逆関数)** of $f$ is the **principal value of the complex logarithm** $f^{-1}(z) = \mathrm{Ln}\, z$.

# 4.2.2 (Complex) Logarithmic Functions (複素)対数関数

Recall that

**Theorem 2.3 Real and Imaginary Parts of a Continuous Function**

Suppose that $f(z) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0$.
Then the complex function (複素関数) $f$ is continuous at the point $z_0$ **if and only if** both real functions (実数値関数) $u$ and $v$ are continuous at the point $(x_0, y_0)$.

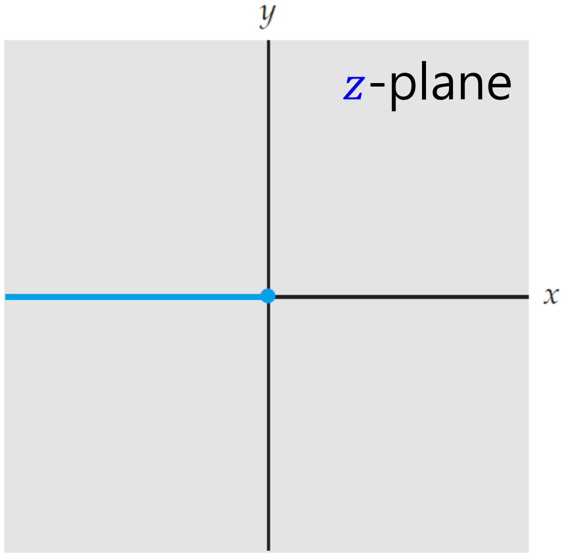



Figure 4.6 $f_1(z)$ defines on domain in gray color excluding blue ray

The principal value of the complex logarithm function

$$\mathrm{Ln}\, z = \log_e |z| + i\,\mathrm{Arg}(z),\ -\pi < \arg(z) \le \pi$$

Real part $u(x,y) = \log_e |z| = \log_e \sqrt{x^2 + y^2}$ is **continuous at all points** in the plane **except** $(0,0)$

Imaginary part $v(x,y) = \mathrm{Arg}(z)$ is **continuous on the domain** $|z| > 0,\ -\pi < \arg(z) < \pi$

Therefore, $\mathrm{Ln}\, z$ is **a continuous function on the domain** $|z| > 0,\ -\pi < \arg(z) < \pi$ (4.1.18)

We give this new function a name by "principal branch of the complex logarithm function"

$$f_1(z) = \log_e |z| + i\,\mathrm{Arg}(z),\ -\pi < \arg(z) < \pi \quad (4.1.19)$$

Here, $f_1(z)$ is $\mathrm{Ln}\, z$ except $\mathrm{Arg}(z) = \pi$

## Theorem 4.4 Analyticity of the Principal Branch of $\ln z$

The principal branch $f_1$ of the complex logarithm defined by (4.1.19) is **an analytic function** and **its derivative is given by**:

$$f_1'(z) = \frac{1}{z} \tag{4.1.20}$$

The theorem 4.4 implies that $\mathrm{Ln}\, z$ is differentiable in the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, and its derivative is given by $f_1'(z)$.

That is, if $|z| > 0$, $-\pi < \arg(z) < \pi$ then

$$\boxed{\frac{d}{dz}\mathrm{Ln}\, z = \frac{1}{z}} \tag{4.1.21}$$

## EXAMPLE (例題) 4.1.5 Derivatives of Logarithmic Functions

Find the derivatives of the function $z\mathrm{Ln}\,z$ in an appropriate domain:

### Solution (解答):

The function $z\mathrm{Ln}\,z$ is differentiable at all points where both of the functions $z$ and $\mathrm{Ln}\,z$ are differentiable.

Because $z$ is entire (整函数) and $\mathrm{Ln}\,z$ is differentiable on the domain given in (4.1.18), as $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows that $z\mathrm{Ln}\,z$ is differentiable on the domain defined by $|z| > 0$, $-\pi < \arg(z) < \pi$

In this domain, the derivative is given by the product rule (積の法則) (3.1.4) of Lecture 3 and (4.1.21):

$$\frac{d}{dz}[\mathrm{zLn}\,z] = z \cdot \frac{1}{z} + 1 \cdot \mathrm{Ln}\,z = 1 + \mathrm{Ln}\,z$$

# Review for Lecture 4

- Harmonic Functions
- (Complex) Exponential Functions
- Exponential Mapping
- (Complex) Logarithmic Functions
- The principal value of the Logarithmic Functions
- Analyticity of the Principal Branch of $\ln z$

# Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

# References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