Lecture 5

5.1 Roots (根) of a Complex Number

5.2 (Complex) Elementary Functions (複素)初等関数 Part 2:

5.2.1 (Complex) Power Functions (複素)冪函数

5.2.2 (Complex) Trigonometric Functions (複素)三角関数 and (Complex) Hyperbolic Functions (複素)双曲線関数

5.1 Roots (根) of a Complex Number

Roots of a Complex Number

Consider to find w in $w^k = z$

where w and z are complex numbers,

k is real, i.e. NOT a complex number.

then

$$w = \sqrt[k]{|z|}\left[\cos\left(\frac{\arg(z) + 2n\pi}{k}\right) + i\sin\left(\frac{\arg(z) + 2n\pi}{k}\right)\right] \qquad (1.4.4)$$

where $n = 0, 1, 2, \ldots, k-1$

For polar form $z = r(\cos\theta + i\sin\theta)$

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta + 2n\pi}{k}\right) + i\sin\left(\frac{\theta + 2n\pi}{k}\right)\right]$$

Quadratic Formula (2次方程式の解の公式)

Suppose a, b, c, x are **real** and $a \neq 0$.

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Two solutions when $b^2 - 4ac \neq 0$

Suppose a, b, c, z are **complex** and $a \neq 0$.

$$az^2 + bz + c = 0 \Rightarrow z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (1.6.3)$$

Still **two solutions** because $\sqrt{b^2 - 4ac}$ represents the set of two square roots of the complex number $b^2 - 4ac$ when it is not 0.

Quadratic Formula (2次方程式の解の公式)

EXAMPLE (例題) 1.6.1 Solve the quadratic equation $z^2+(1-i)z-3i=0$.

Solution (解答):

From Equation (1.6.3), with $a=1$, $b=1-i$, and $c=-3i$ we have

$$z=\frac{-(1-i)+\sqrt{(1-i)^2-4\cdot 1\cdot(-3i)}}{2}=\frac{1}{2}\left[-1+i+\sqrt{10i}\right]$$

To compute $\sqrt{10i}=(10i)^{\frac{1}{2}}$, we follow the Equation (1.4.4) in this Lecture, and obtain

$\left|(10i)^{\frac{1}{2}}\right|=\sqrt{10}$, $\arg\left((10i)^{\frac{1}{2}}\right)=\frac{\pi}{2}$, $k=2$, $n=0$ and $n=1$ (because $n\le k-1$).

Therefore, two square roots of $10i$ are

$$s_{n=0}=\sqrt{10}\left(\cos\frac{\frac{\pi}{2}+2\cdot 0\cdot\pi}{2}+i\sin\frac{\frac{\pi}{2}+2\cdot 0\cdot\pi}{2}\right)=\sqrt{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{10}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{5}+i\sqrt{5}$$

$$s_{n=1}=\sqrt{10}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\sqrt{10}\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=-\sqrt{5}-i\sqrt{5}$$

Quadratic Formula (2次方程式の解の公式)

Solution (解答) (cont.):

Therefore, (1.4.4) gives two values:

$$z_1 = \frac{1}{2}\left[-1 + i + \left(\sqrt{5} + i\sqrt{5}\right)\right]$$

$$= \frac{1}{2}\left(\sqrt{5} - 1\right) + \frac{1}{2}\left(\sqrt{5} + 1\right)i$$

and

$$z_2 = \frac{1}{2}\left[-1 + i + \left(-\sqrt{5} - i\sqrt{5}\right)\right]$$

$$= -\frac{1}{2}\left(\sqrt{5} + 1\right) - \frac{1}{2}\left(\sqrt{5} - 1\right)i$$

5.2 (Complex) Elementary Functions (複素初等関数) Part 2:

5.2.1 (Complex) Power Functions (複素冪函数)

Complex Power Function (複素冪函数)

Suppose we know $z = e^{\ln z}$, for all nonzero complex numbers z,

(Because $\ln z = \ln e^{\ln z} = \ln z$)

From Theorem 4.2 (iv) in Lecture 4, $z^n = \left(e^{\ln z}\right)^n = e^{n \ln z}$

Definition 4.4 Complex Power Function (複素冪函数)

If α is a complex number and $z \neq 0$, then the Complex Power Function (複素冪函数) is defined to be:

$$z^{\alpha} = e^{\alpha \ln z} \tag{4.2.1}$$

EXAMPLE (例題) 4.2.1 Complex Power Function

Find the values of the given complex power: (a) i^{2i} (b) $(1+i)^i$

Solution (解答):

(a) In part (a) of Example 4.1.3 in Lecture 4, we know

$$\ln i = \frac{(4n+1)\pi}{2} i \quad \text{, for } n = 0, \pm 1, \pm 2, \ldots$$

Thus, for i^{2i}, by identifying $z = i$ and $\alpha = 2i$ in Equation (4.2.1) we obtain:

$$i^{2i} = e^{2i \ln i} = e^{2i \cdot \frac{(4n+1)\pi}{2} i} = e^{-(4n+1)\pi} \quad \text{, for } n = 0, \pm 1, \pm 2, \ldots$$

For example, when $n = -1$, 0, and 1, the values of i^{2i} are 12391.6, 0.0432, and 1.507×10^{-7}, respectively.

Solution (解答)(cont.):

(**b**) From part (b) of Example 4.1.3 in Lecture 4, we know

$$\ln(1+i) = \frac{1}{2}\log_e 2 + \frac{(8n+1)\pi}{4} i \quad , \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Thus, for $(1+i)^i$, by identifying $z = 1 + i$ and $\alpha = i$ in Equation (4.2.1) we obtain:

$$(1+i)^i = e^{i\ln(1+i)} = e^{i\cdot\left[\frac{1}{2}\log_e 2 + \frac{(8n+1)\pi}{4} i\right]} \quad , \text{for } n = 0, \pm 1, \pm 2, \ldots$$

$$= e^{-\frac{(8n+1)\pi}{4} + \frac{i}{2}\log_e 2} \quad , \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Properties (性質) of Complex Power Functions

Complex powers defined by (4.2.1) satisfy the following properties that are analogous to (類似する) properties of real powers:

(i) $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1+\alpha_2}$

(ii) $\dfrac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1-\alpha_2}$ (4.2.5)

(iii) $(z^{\alpha})^{n} = z^{n\alpha}, n = 0, \pm 1, \pm 2, \ldots$

Definition 4.5 Principal Value (主値) of a Complex Power Function

If α is a complex number and $z \neq 0$, then the function defined by

$$z^{\alpha} = e^{\alpha \operatorname{Ln} z} \qquad (4.2.6)$$

is called **the** principal value of the complex power function z^{α}.

EXAMPLE (例題) 4.2.2 Principal Value of a Complex Power Function

Find the principal value of each complex power: (a) $(-3)^{\frac{i}{\pi}}$ (b) $(2i)^{1-i}$

Solution (解答):

(a) For $z = -3$, we have $|z| = 3$ and $\mathrm{Arg}(-3) = \pi$, and so $\mathrm{Ln}\,(-3) = \log_e 3 + i\pi$ by Equation (4.1.14) in Lecture 4.

Thus, by identifying $z = -3$ and $\alpha = \frac{i}{\pi}$ in (4.2.6), we obtain:

$$(-3)^{\frac{i}{\pi}} = e^{\frac{i}{\pi}\mathrm{Ln}\,(-3)} = e^{\frac{i}{\pi}(\log_e 3 + i\pi)} = e^{\frac{i\log_e 3}{\pi} - 1} = e^{\frac{i\log_e 3}{\pi}} e^{-1}$$

$$= \left(\cos\frac{\log_e 3}{\pi} + i\sin\frac{\log_e 3}{\pi}\right) e^{-1}$$

$$\approx 0.3456 + 0.1260i$$

Not necessary for the assignment report. (same for all others)

Solution (解答)(cont.):

(b) For $z = 2i$, we have $|z| = 2$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$, and so $\mathrm{Ln}\,(2i) = \log_e 2 + i\,\frac{\pi}{2}$

by (4.1.14) in Lecture 4.

Thus, by identifying $z = 2i$ and $\alpha = 1 - i$ in (4.2.6), we obtain:

$$(2i)^{1-i} = e^{(1-i)\,\mathrm{Ln}\,2i} = e^{(1-i)\left(\log_e 2 + i\frac{\pi}{2}\right)} = e^{\log_e 2 + \frac{\pi}{2} - i\left(\log_e 2 - \frac{\pi}{2}\right)}$$

$$= e^{\log_e 2 + \frac{\pi}{2}} e^{-i\left(\log_e 2 - \frac{\pi}{2}\right)}$$

$$= e^{\log_e 2 + \frac{\pi}{2}}\left[\cos\left(\log_e 2 - \frac{\pi}{2}\right) - i\sin\left(\log_e 2 - \frac{\pi}{2}\right)\right]$$

$$\approx 6.1474 + 7.4008i$$

Analyticity (解析性) of z^α

Since the function $e^{\alpha z}$ is **continuous on** the **entire complex plane**, and since the function $\mathrm{Ln}\, z$ is **continuous on** the domain $|z| > 0, -\pi < \arg(z) < \pi$, it follows that z^α is **continuous on** the domain $|z| > 0, -\pi < \arg(z) < \pi$.

$$f_1(z) = e^{\alpha\, \mathrm{Ln}\, z} = e^{\alpha(\log_e |z| + i \arg(z))}, \quad -\pi < \arg(z) < \pi \qquad (4.2.7)$$

This is called the **principal branch of the complex power** z^α.

Analyticity (解析性) of z^α

The derivative of $f_1(z)$ can be found using the chain rule (3.1.6) in Lecture 3,

$$f_1'(z) = \frac{d}{dz} e^{\alpha \operatorname{Ln} z} = e^{\alpha \operatorname{Ln} z} \frac{d}{dz}[\alpha \operatorname{Ln} z] = e^{\alpha \operatorname{Ln} z} \frac{\alpha}{z} \qquad (4.2.8)$$

Using the **principal value** $z^\alpha = e^{\alpha \operatorname{Ln} z}$ we find that (4.2.8) simplifies to

$$f_1'(z) = z^\alpha \frac{\alpha}{z} = \alpha z^{\alpha - 1}$$

On the domain $|z| > 0, -\pi < \arg(z) < \pi$, the **principal value of the complex power** z^α is **differentiable** and

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}, \text{ where } \alpha \text{ is a complex number.} \qquad (4.2.9)$$

EXAMPLE (例題) 4.2.3 Derivative of a Complex Power Function

Find the derivative of the principal value z^i at the point $z = 1 + i$.

Solution (解答):

Because the point $z = 1 + i$ is in the domain $|z| > 0, -\pi < \arg(z) < \pi$, then

from Equation (4.2.9) $\frac{d}{dz} z^i = iz^{i-1}$ we have

$$\left.\frac{d}{dz} z^i\right|_{z=1+i} = \left. iz^{i-1}\right|_{z=1+i} = i(1+i)^{i-1}$$

We can use (4.2.5) $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$ to rewrite this value as:

$$i(1+i)^{i-1} = i(1+i)^i (1+i)^{-1} = i(1+i)^i \frac{1}{1+i}$$

$$= \frac{i(1-i)}{(1+i)(1-i)} (1+i)^i = \frac{1+i}{2} (1+i)^i$$

Solution (解答)(cont.):

Moreover, from part (b) of Example 4.2.1 we have

$$(1+i)^i = e^{i\ln(1+i)} = e^{i\cdot\left[\frac{1}{2}\log_e 2+\frac{(8n+1)\pi}{4}i\right]}, \text{ for } n = 0, \pm 1, \pm 2, \dots$$

We ask for the **principal value** of $(1+i)^i$ for $n = 0$ because $-\pi < \arg(z) < \pi$, then

$$(1+i)^i = e^{-\frac{\pi}{4}+\frac{i}{2}\log_e 2}$$

Above all, we have

$$\left.\frac{d}{dz}z^i\right|_{z=1+i} = \frac{1+i}{2}(1+i)^i = \frac{1+i}{2}e^{-\frac{\pi}{4}+\frac{i}{2}\log_e 2} = \frac{1+i}{2}e^{-\frac{\pi}{4}}e^{\frac{i}{2}\log_e 2}$$

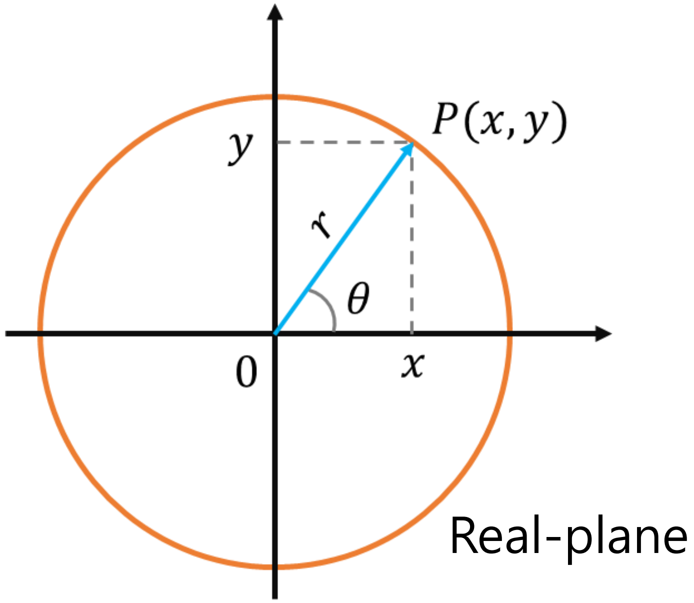
$$\approx 0.1370 + 0.2919i$$

5.2 (Complex) Elementary Functions (複素初等関数) Part 2:

5.2.2 (Complex) Trigonometric Functions

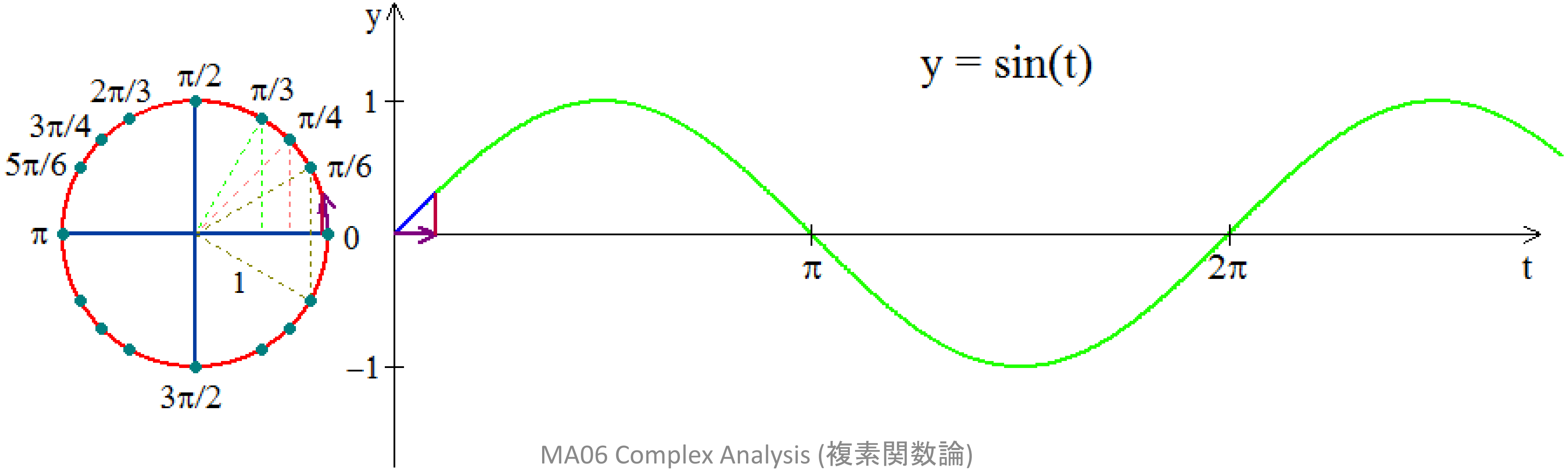
(複素三角関数)

Recall Real Trigonometric Functions (実三角関数)



$$\sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x} \qquad \sec\theta = \frac{r}{x}$$

$$\cos\theta = \frac{x}{r} \qquad \cot\theta = \frac{x}{y} \qquad \csc\theta = \frac{r}{y}$$



5.2.2 (Complex) Trigonometric Functions (複素)三角関数

Recall that in **MA04** ***Calculus II (微積分 II)***,

we have **Taylor Series** (テイラー級数) :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

5.2.2 (Complex) Trigonometric Functions (複素)三角関数

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$ where x is a real number

Replace x with ix, then

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$ because $i^2 = -1$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

i.e. $e^{ix} = \cos x + i \sin x$ Euler's Formula !

5.2.2 (Complex) Trigonometric Functions (複素)三角関数

Leonhard Euler
レオンハルト・オイラー
(Switzerland) (1707~1783)

① $e^{ix} = \cos x + i \sin x$ **Euler's Formula**

② $e^{-ix} = \cos(-x) + i \sin(-x)$
$= \cos x - i \sin x$ (4.3.1)

① + ② $e^{ix} + e^{-ix} = 2\cos x$ ⇨ $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ (4.3.2)

① − ② $e^{ix} - e^{-ix} = i2\sin x$ ⇨ $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ (4.3.3)

Definition 4.6 Complex Sine and Cosine Functions (複素正弦関数と複素余弦関数)

The complex sine and cosine functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad (4.3.4)$$

Analogous to (類似する) real trigonometric functions (実三角関数), we can define the complex tangent (複素正接), cotangent (複素余接), secant (複素正割), and cosecant (複素余割) functions by using the complex sine (複素正弦) and cosine (複素余弦):

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z} \quad \text{and} \quad \csc z = \frac{1}{\sin z} \qquad (4.3.5)$$

EXAMPLE (例題) 4.3.1 Values of Complex Trigonometric Functions

Express the value of the given trigonometric function in the form $a + ib$.

(a) $\cos i$ (b) $\sin(2+i)$ (c) $\tan(\pi - 2i)$

Solution (解答):

(a) By Equation (4.3.4),

$$\cos i = \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \frac{e^{-1} + e}{2} \approx 1.5431$$

Not necessary for the assignment report.
(same for all others)

(b) By Equation (4.3.4),

$$\sin(2+i) = \frac{e^{i(2+i)} - e^{-i(2+i)}}{2i} = \frac{e^{-1+2i} - e^{1-2i}}{2i} = \frac{e^{-1}e^{2i} - e\cdot e^{-2i}}{2i}$$

$$= \frac{e^{-1}(\cos 2 + i\sin 2) - e(\cos(-2) + i\sin(-2))}{2i}$$

$$\approx 1.4031 - 0.4891i$$

Solution (解答)(cont.):

(c) By Equation (4.3.5) and (4.3.4),

$$\tan(\pi - 2i) = \frac{\sin(\pi - 2i)}{\cos(\pi - 2i)} = \frac{\frac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{2i}}{\frac{e^{i(\pi-2i)} + e^{-i(\pi-2i)}}{2}}$$

$$= \frac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{\left(e^{i(\pi-2i)} + e^{-i(\pi-2i)}\right)i} = \frac{\left(e^{2}e^{i\pi} - e^{-2}e^{-i\pi}\right)i}{\left(e^{2}e^{i\pi} + e^{-2}e^{-i\pi}\right)i \cdot i}$$

$$= -\frac{\left(e^{2}(\cos\pi + i\sin\pi) - e^{-2}(\cos\pi - i\sin\pi)\right)i}{\left(e^{2}(\cos\pi + i\sin\pi) + e^{-2}(\cos\pi - i\sin\pi)\right)}$$

$$= -\frac{-(e^{2} - e^{-2})i}{-(e^{2} + e^{-2})} \approx -0.9640i$$

Complex Trigonometric Identities (複素三角恒等式)

(i) $\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$ (4.3.6)

(ii) $\sin^2 z + \cos^2 z = 1$ (4.3.7)

(iii) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ (4.3.8)

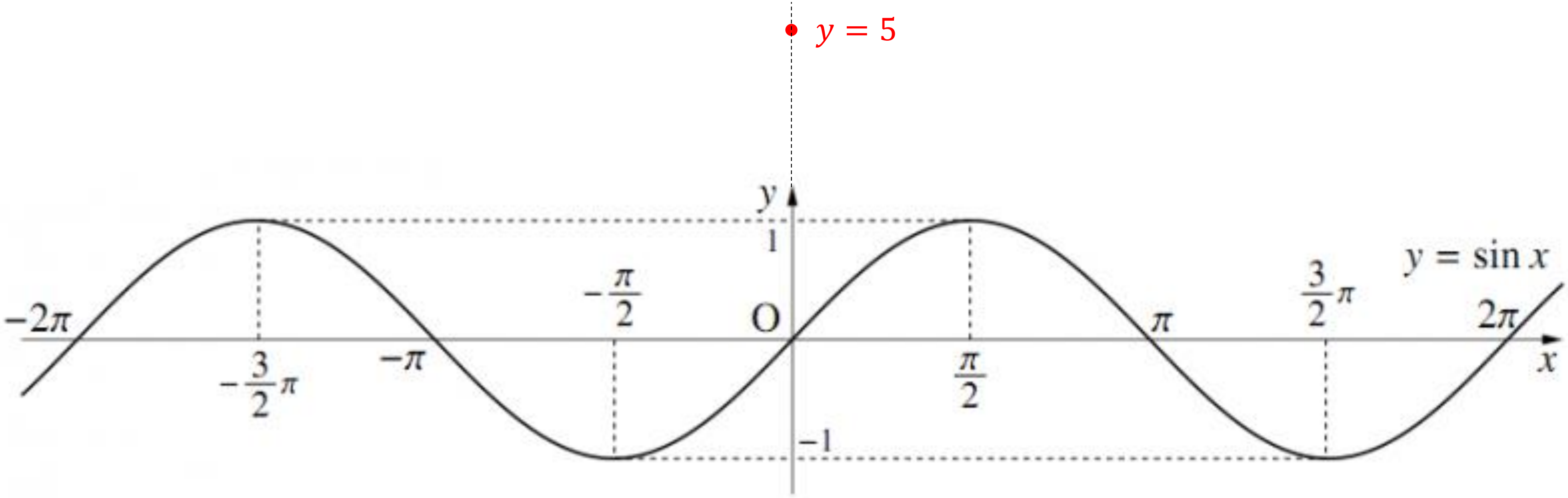
(iv) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$ (4.3.9)

(v) $\sin 2z = 2 \sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$ (4.3.10)

(vi) $\sin(z + 2\pi) = \sin z \qquad \cos(z + 2\pi) = \cos z$ Periodicity (周期性) (4.3.11)

5.2.2 (Complex) Trigonometric Functions (複素)三角関数

Recall the $\sin x$, where $x \in \mathbf{R}$ (i.e. x is a real number (実数).)



Notice: We cannot find a solution for $\sin x = 5$,

when $x \in \mathbf{R}$, because $|\sin x| \leq 1$.

EXAMPLE (例題) 4.3.2 Solving Complex Trigonometric Equations
Find all solutions to the equation $\sin z = 5$, where $z \in \mathbf{C}$.

Solution (解答):

By Definition 4.6,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 5$$

Multiplying this equation by e^{iz}

$$\frac{e^{iz}\left(e^{iz} - e^{-iz}\right)}{2i} = 5e^{iz}$$

$$e^{iz+iz} - e^{iz-iz} = 10ie^{iz}$$

$$e^{i2z} - e^{0} = 10ie^{iz}$$

$$e^{i2z} - 10ie^{iz} - 1 = 0$$

$$\left(e^{iz}\right)^2 - 10i\left(e^{iz}\right) - 1 = 0$$

From the quadratic formula (1.6.3) in this Lecture that the solutions of $az^2 + bz + cz = 0$ are given by

$$e^{iz} = \frac{-(-10i) + \sqrt{(-10i)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{10i + \sqrt{-96}}{2}$$

$$= 5i \pm 2\sqrt{6}i$$

$$= \left(5 \pm 2\sqrt{6}\right)i$$

Now we need to solve $e^{iz} = \left(5 \pm 2\sqrt{6}\right)i$

Solution (解答)(cont.):

- For $e^{iz} = (5 + 2\sqrt{6})i$

$$iz = \ln\left[(5 + 2\sqrt{6})i\right]$$

$$z = -i\ln\left[(5 + 2\sqrt{6})i\right]$$

$$= -i\left[\log_e\left|(5 + 2\sqrt{6})i\right| + i\arg\left[(5 + 2\sqrt{6})i\right]\right]$$

Because $(5 + 2\sqrt{6})i$ is a pure imaginary number (純虚数) and $5 + 2\sqrt{6} > 0$, we have $\arg\left[(5 + 2\sqrt{6})i\right] = \frac{\pi}{2} + 2n\pi$

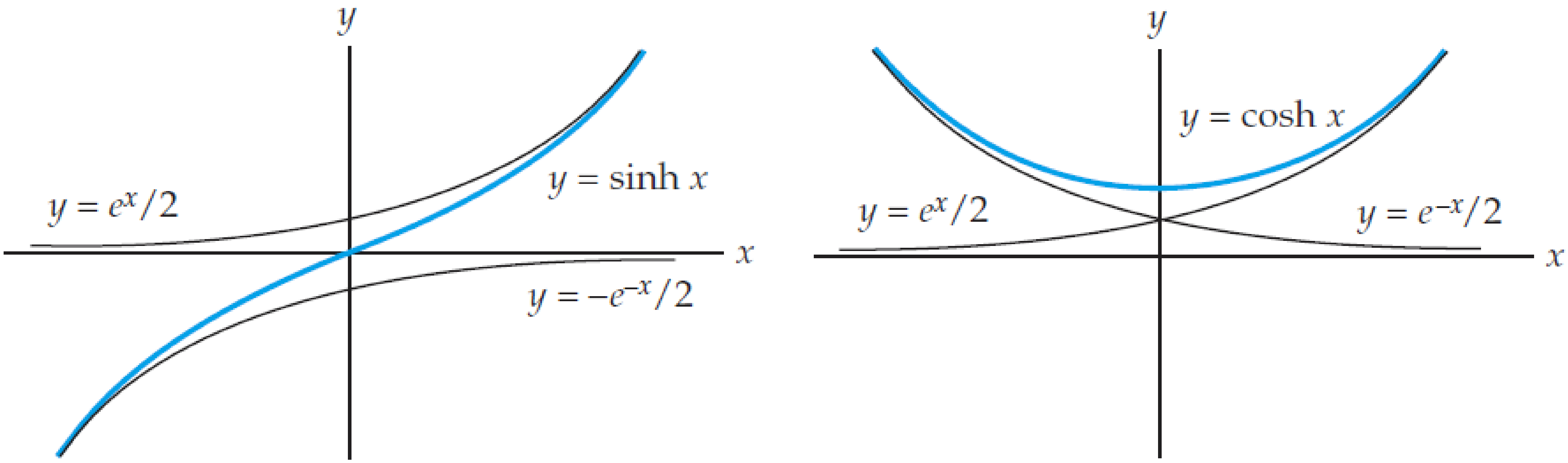
$$= -i\left[\log_e(5 + 2\sqrt{6}) + i\left(\frac{\pi}{2} + 2n\pi\right)\right] \quad \text{for } n = 0, \pm1, \pm2, \ldots$$

$$= \frac{(4n + 1)\pi}{2} - i\log_e(5 + 2\sqrt{6}) \quad \text{for } n = 0, \pm1, \pm2, \ldots$$

- For $e^{iz} = (5 - 2\sqrt{6})i$, similarly we have

$$z = \frac{(4n + 1)\pi}{2} - i\log_e(5 - 2\sqrt{6}) \quad \text{for } n = 0, \pm1, \pm2, \ldots$$

Recall the real hyperbolic functions (実双曲線関数).



(a) $y = \sinh x = \frac{e^{x}-e^{-x}}{2}$ (b) $y = \cosh x = \frac{e^{x}+e^{-x}}{2}$

Figure 4.11 The real hyperbolic functions, $x \in \mathbf{R}$

Modulus (複素数の絶対値) of Sine and Cosine Functions

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$= \sin x \cosh y + i\cos x \sinh y$$ where $z = x + iy$ (4.3.16)

$$\cos z = \cos x \cosh y - i\sin x \sinh y$$ (4.3.17)

$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \sqrt{\sin^2 x\,(1 + \sinh^2 y) + \cos^2 x \sinh^2 y}$$ Because $\cosh^2 y = 1 + \sinh^2 y$

$$= \sqrt{\sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y}$$

$$= \sqrt{\sin^2 x + \sinh^2 y}$$ Because $\sin^2 x + \cos^2 x = 1$ (4.3.18)

$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$$ (4.3.19)

Zeros

$\sin z = 0$ if and only if $z = n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

$\cos z = 0$ if and only if $z = \frac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$

EXAMPLE (例題) **Find the solution to the equation $\sin z = 0$.**

Solution (解答):

- Method 1: Use the similar way in Example 4.3.2.
- Method 2: Recall the Zero in Lecture 1, a complex number is equal to 0 if and only if its modulus is 0, then

$$|\sin z| = 0$$

$$\sqrt{\sin^2 x + \sinh^2 y} = 0 \Rightarrow \begin{cases} \sin^2 x = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi, \text{ for } n = 0, \pm 1, \pm 2, \ldots \\ \sinh^2 y = 0 \Rightarrow \sinh y = 0 \Rightarrow y = 0 \end{cases}$$

According to the Figure 4.11(a)

Therefore, $z = x + iy = n\pi + i0 = n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$

Derivatives of Complex Trigonometric Functions

$$\frac{d}{dx}\sin z = \cos z \qquad \frac{d}{dx}\cos z = -\sin z$$

$$\frac{d}{dx}\tan z = \sec^2 z \qquad \frac{d}{dx}\cot z = -\csc^2 z$$

$$\frac{d}{dx}\sec z = \sec z \tan z \qquad \frac{d}{dx}\csc z = -\csc z \cot z$$

Proof for $\frac{d}{dz}\sin z$ **:** $$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Analyticity

- The $\sin z$ and $\cos z$ are **entire (整函数)**.
- But $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are **only analytic (解析的) at those points where the denominator (分母) is nonzero (非ゼロ)**.

Review for Lecture 5

- Roots (根) of a Complex Number
- Quadratic Formula (2次方程式の解の公式)
- (Complex) Power Functions (複素)冪函数
- (Complex) Trigonometric Functions (複素)三角関数
- (Complex) Hyperbolic Functions (複素)双曲線関数

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Elementary function: https://en.wikipedia.org/wiki/Elementary_function

[3] 複素三角関数 ～ 単位円の束縛を超えて http://taketo1024.hateblo.jp/entry/complex-trigonometric

*Trigonometric Mapping