Lecture 6

*6.1 Review of Real Line Integral (実·線積分)

6.2 Complex Integral (複素積分)

Real Line Integral

(実·線積分)

in the Cartesian Plane

Complex Integral

(複素積分)

in the Complex Plane

*6.1 Review of Real Line Integral

(実·線積分)

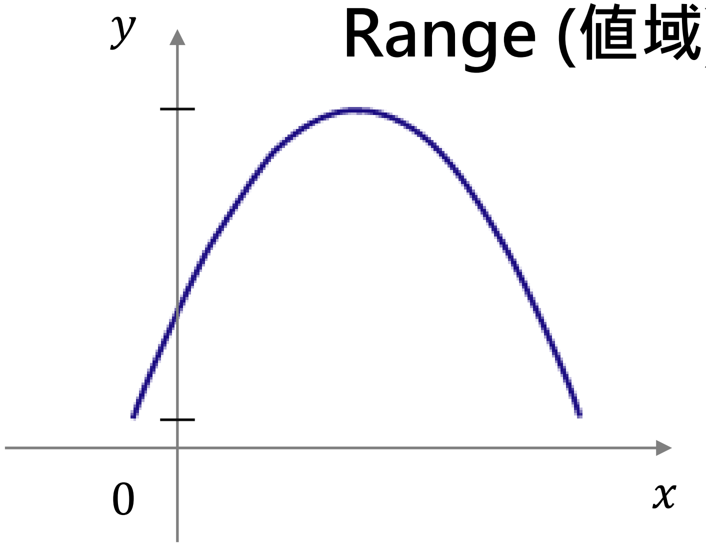
Notice: In all lecture notes, the contents marked with * are not in the scope of the final examination.

*6.1 Review of Real Line Integral (実·線積分)

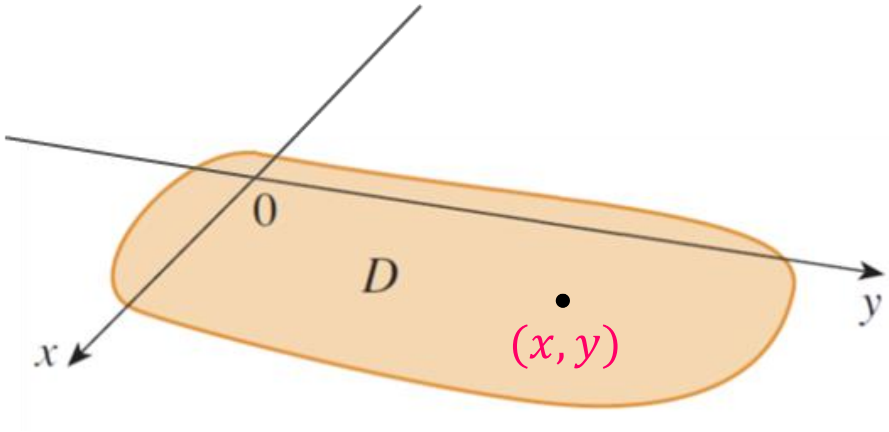
Domain (定義域) $x \in \mathbf{R}$

$(x, f(x))$

Range (值域)

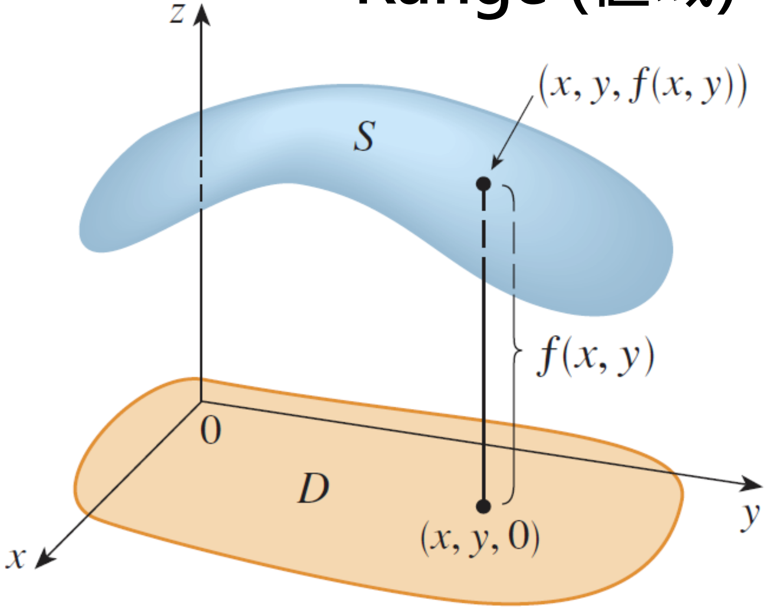


Domain (定義域) $(x, y) \in \mathbf{D}$



$(x, y, f(x, y))$

Range (值域)



*6.1 Review of Real Line Integral (実·線積分)

One-Variable Calculus -- Definite integral (定積分) of f

$$\int_a^b f(x)dx = \lim_{\|\Delta x\| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{\|\Delta x\| \to 0} [f(x_k^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

One-Variable Function

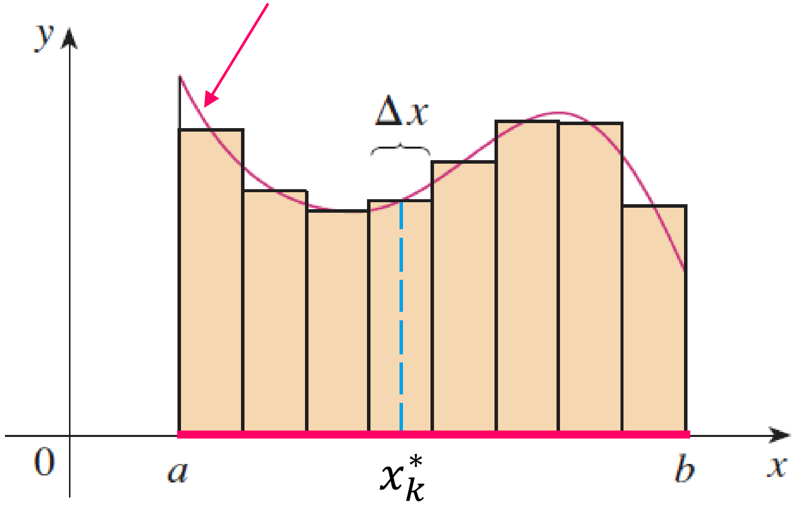
relationship, called **function**

$$y = f(x)$$

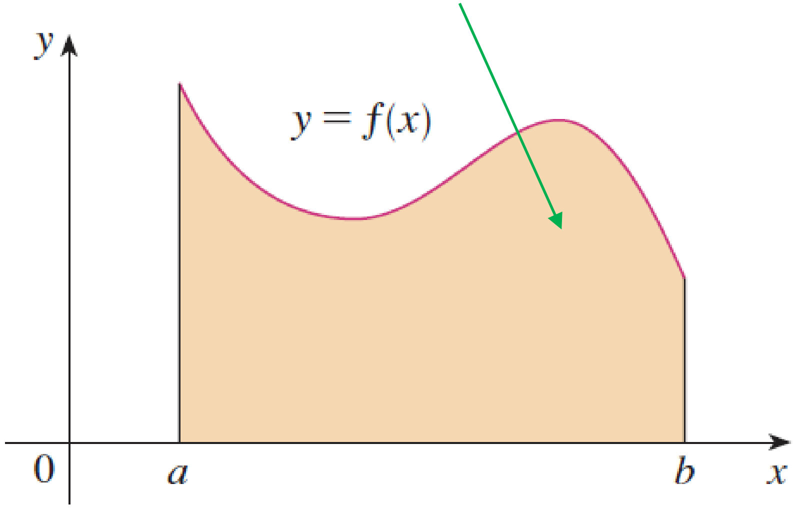
Independent variable

dependent variable

$$y = f(x)$$



$$Area(面積) = \int_a^b f(x)dx$$



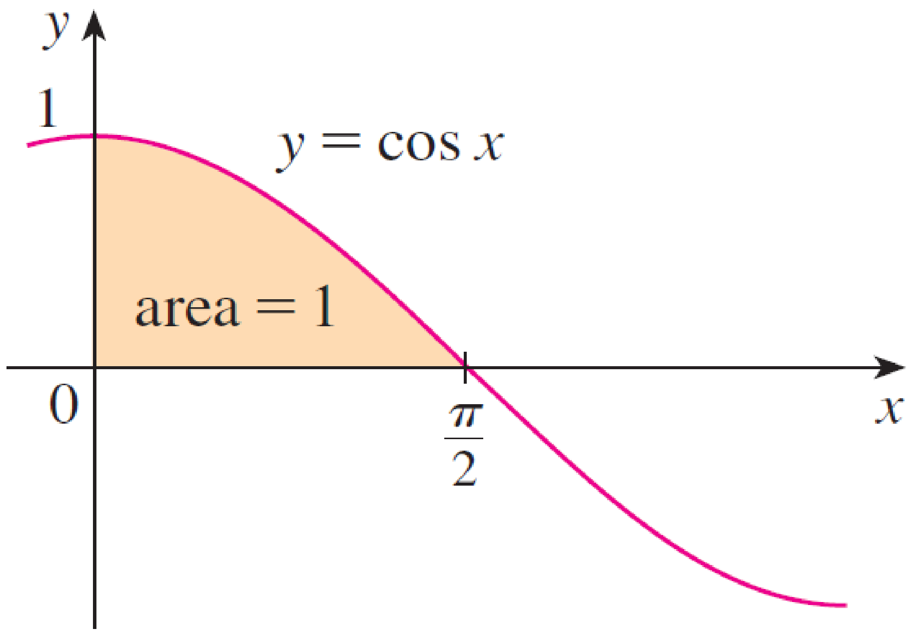
$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \Delta x,$$

i.e. equal interval (等間隔)

*6.1 Review of Real Line Integral (実·線積分)

EXAMPLE (例題) 1. Recall Definite integral (定積分) in *Calculus I (微積分 I)*

Evaluate the integral $\int_0^{\frac{\pi}{2}} \cos x \, dx$

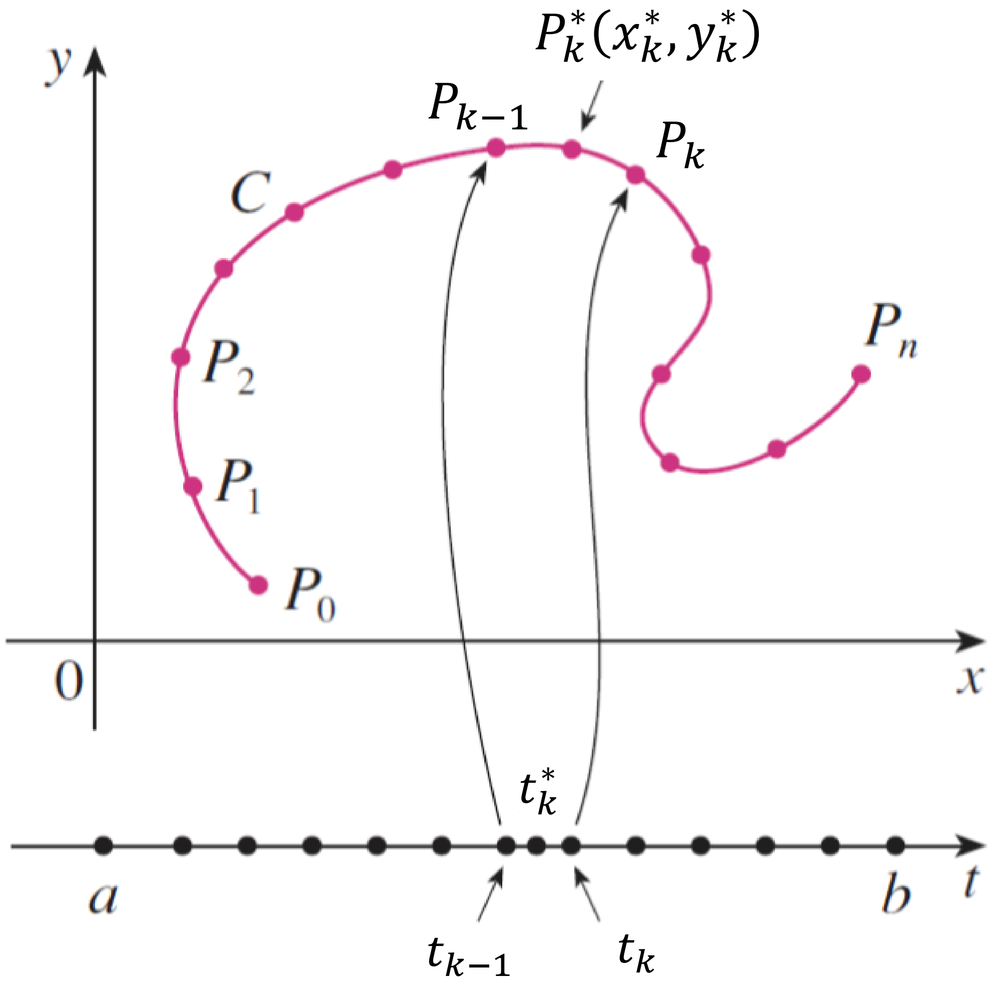


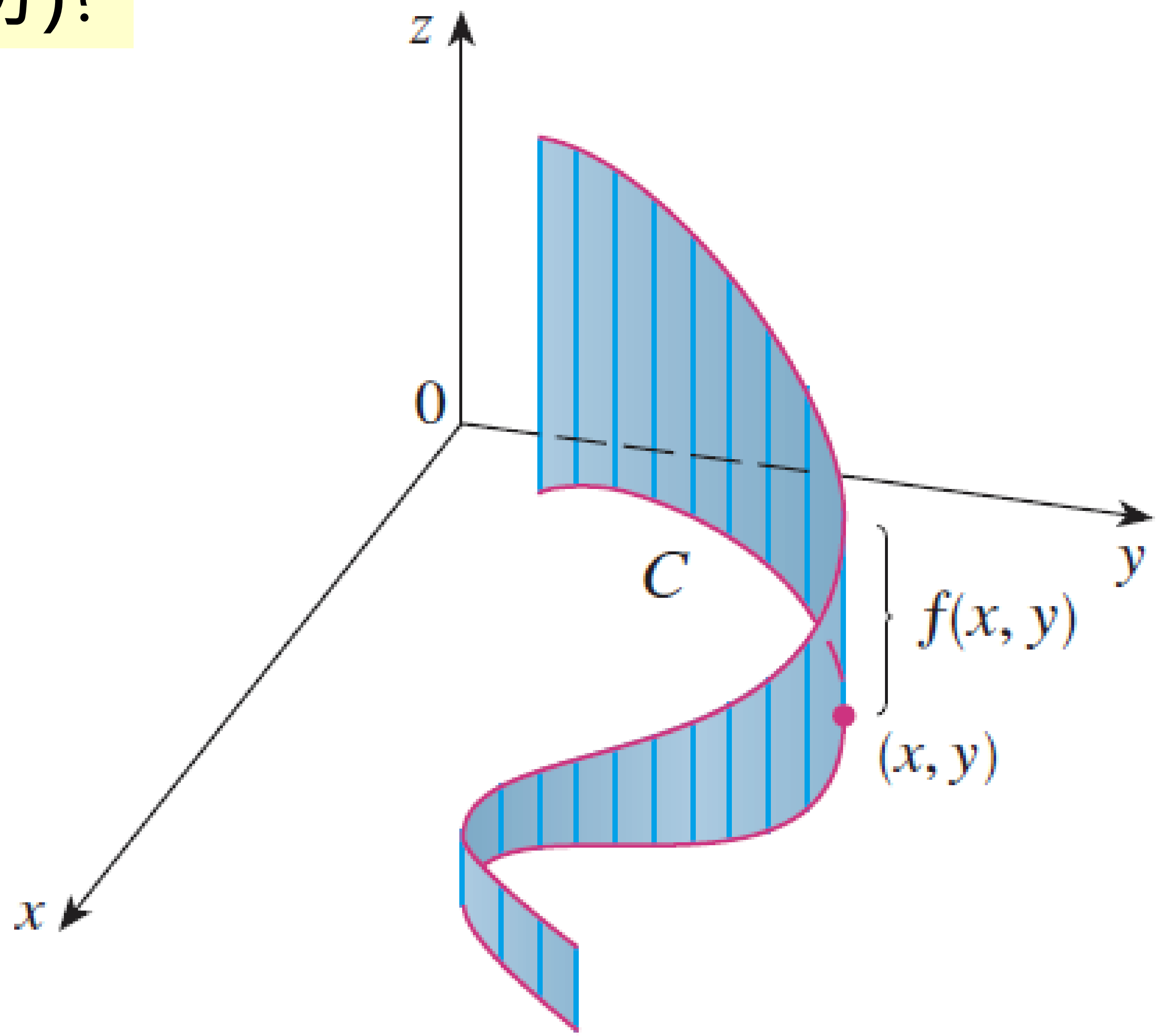
Solution (解答):

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1$$

*6.1 Review of Real Line Integral (実·線積分)

Recall: What is **Line integral (線積分)**?



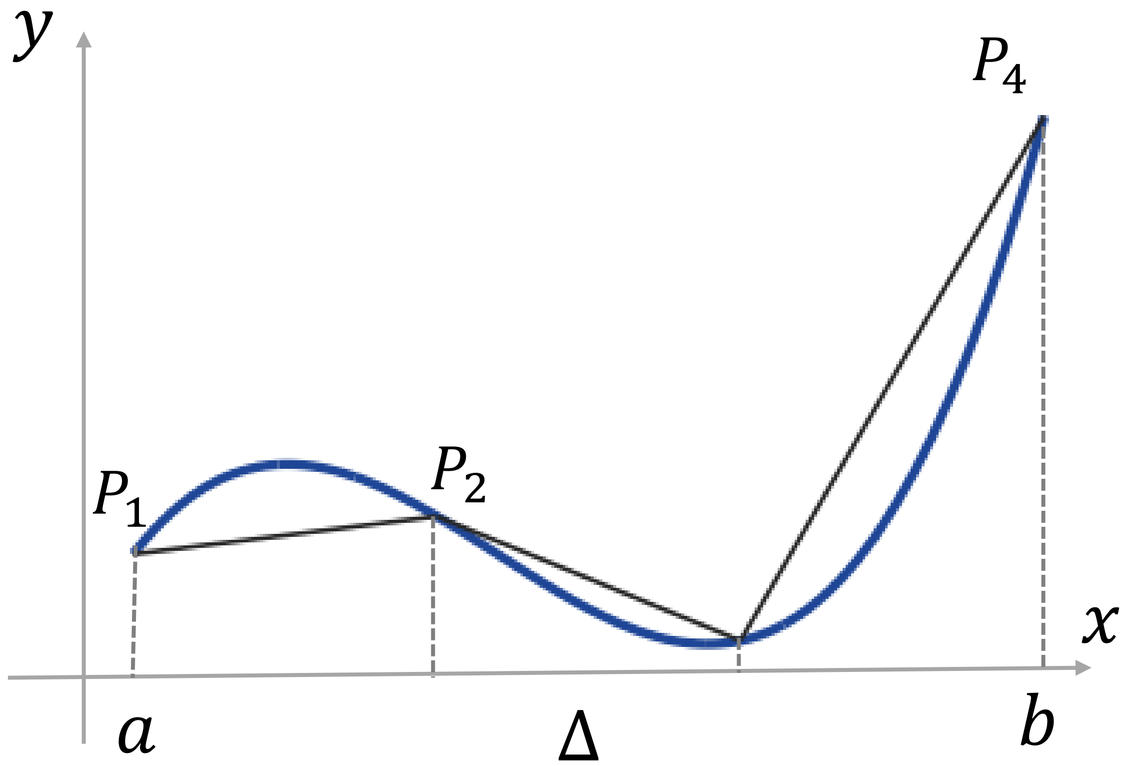


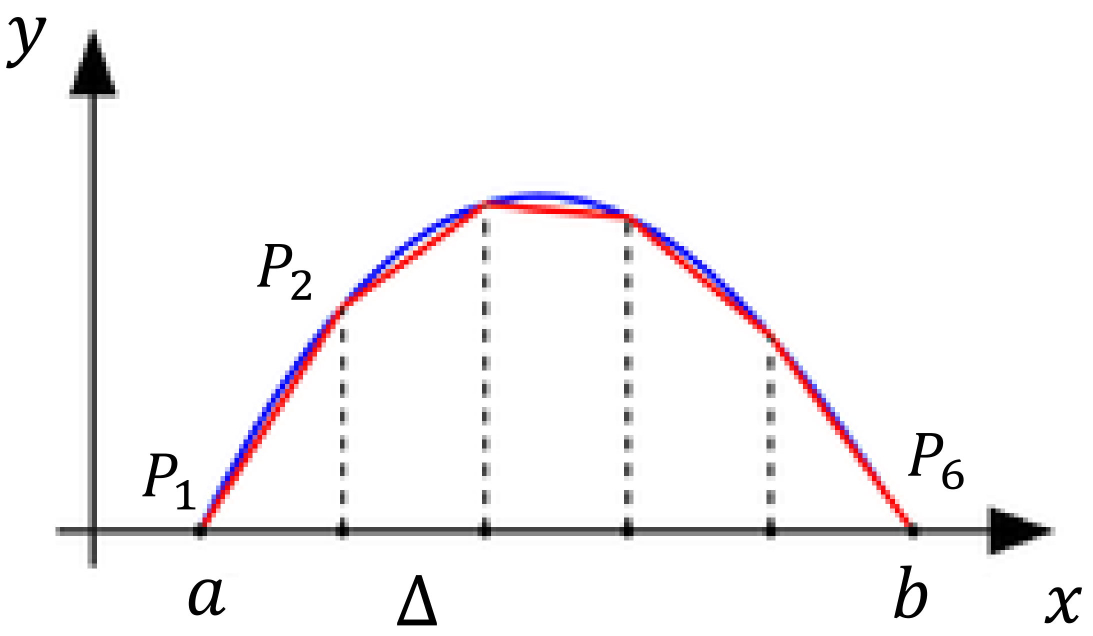
*6.1 Review of Real Line Integral (実·線積分)

Arc Length (弧長)

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$





*6.1 Review of Real Line Integral (実·線積分)

Line Integral (線積分)

If f is defined on a **smooth** (滑らか) or **piecewise-smooth** (区分的滑らか) **curve** C, then **the line integral of f along C** is

$$\int_C f(x,y)ds = \lim_{\|\Delta s_{max}\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k$$

if this limits exists. (Here the norm $\|\Delta s_{max}\|$ defines the length of the longest subinterval (部分区間))

How to compute Line Integral?

By introducing **Arc length (弧長)** $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, we have

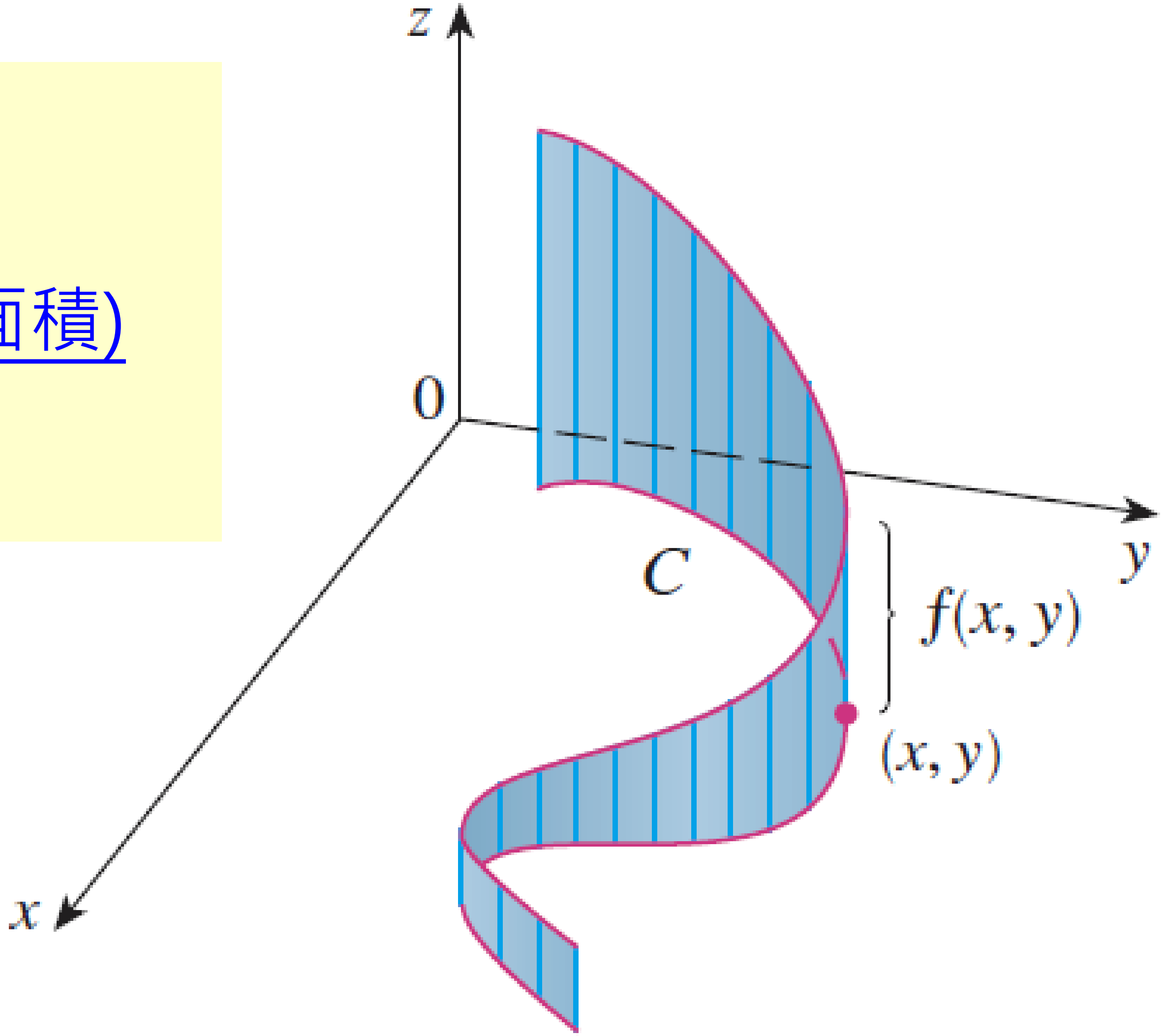
$$\int_C f(x,y)ds = \int_C f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

*6.1 Review of Real Line Integral (実·線積分)

Insight (洞察) of Line Integrals

In fact, if $f(x, y) > 0$,

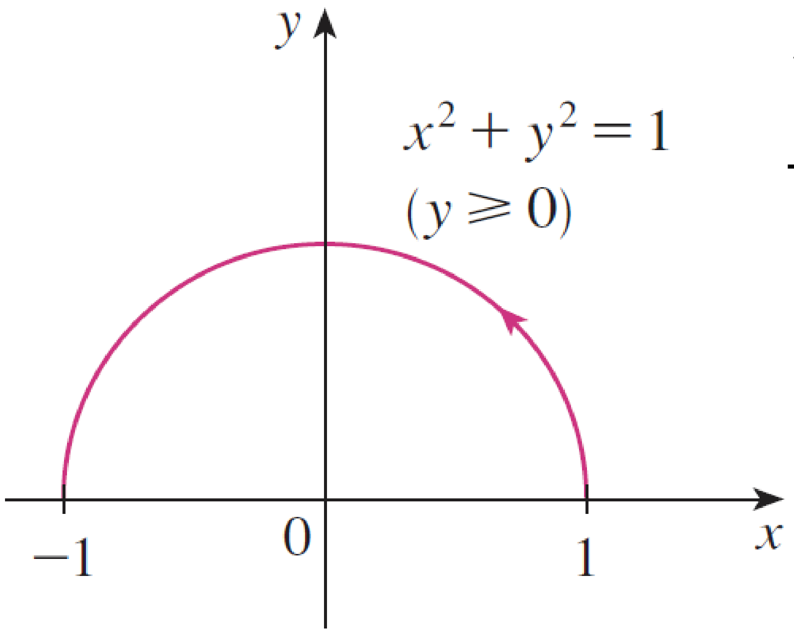
$\int_c f(x, y)ds$ represents the area (面積) of **one side** of the "curtain".



EXAMPLE (例題) 2. Recall Real Line Integral in *Calculus II (微積分 II)*

Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$

Solution (解答):



Recall that the unit circle can be parametrized by

$$x = \cos t, \quad y = \sin t$$

And the upper half of the circle is described by the parameter interval $0 \le t \le \pi$

Therefore, from the formula in Page 8 of this lecture note, we have

$$\int_C (2 + x^2 y) ds = \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

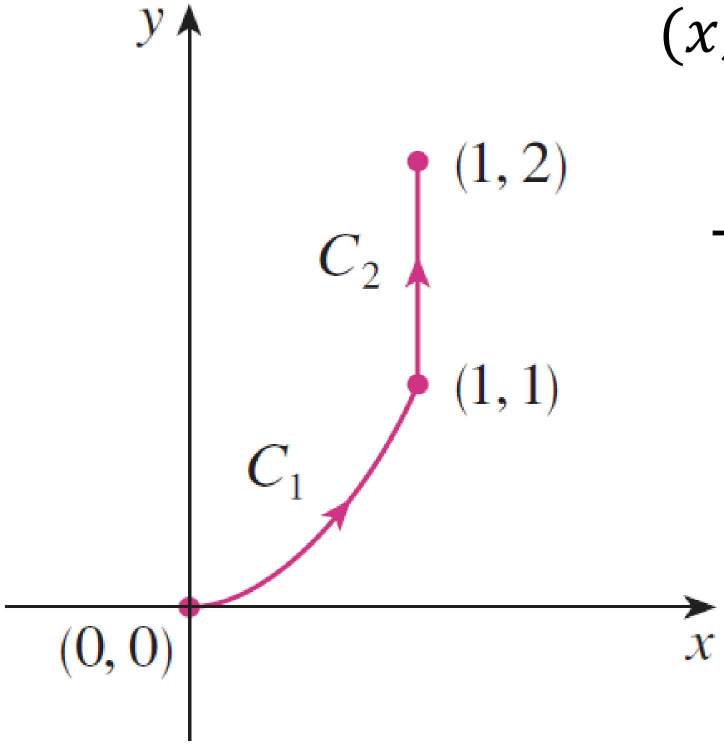
$$= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{(-\sin t)^2 + \cos^2 t}\, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t) = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi = 2\pi + \frac{2}{3}$$

EXAMPLE (例題) 3. Recall Real Line Integral in *Calculus II (微積分 II)*

Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution (解答): ① Because from C_1 we know y is a function of x, i.e. the domain (x, y) becomes (x, x^2), so we can use x as the parameter t, then



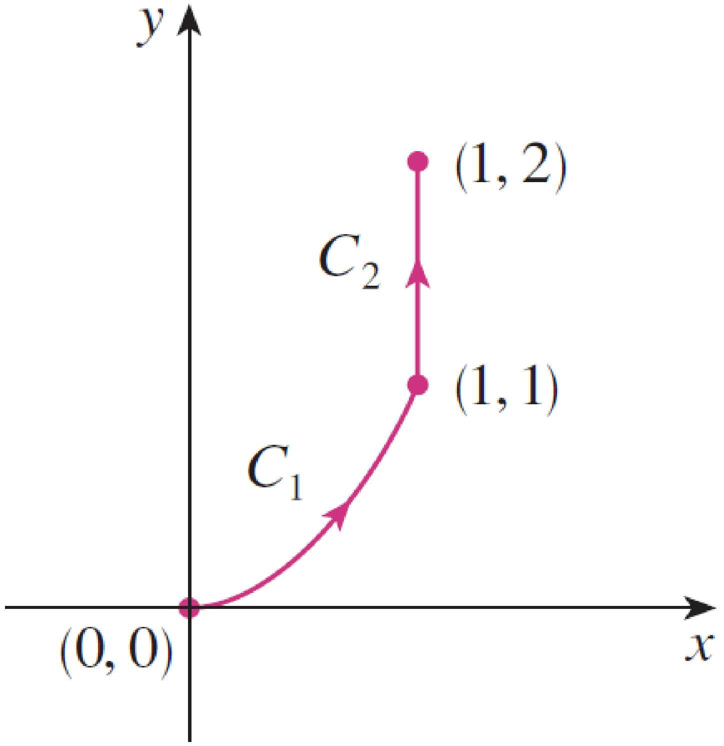
$$x = x, \quad y = x^2, \quad 0 \le x \le 1$$

Therefore

$$\int_{C_1} 2x ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2x\sqrt{1 + (2x)^2} dx$$

$$= \left[\frac{1}{4}\cdot\frac{2}{3}(1 + 4x^2)^{\frac{3}{2}}\right]_0^1 = \frac{5\sqrt{5} - 1}{6}$$

Solution (解答)(cont.):



② Because from C_2 we see a vertical line segment, so we can use y as the parameter t, then

$$x = 1, \quad y = y, \quad 1 \leq y \leq 2$$

Therefore

$$\int_{C_2} 2x ds = \int_1^2 2 \cdot 1 \cdot \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy$$

$$= \int_1^2 2\sqrt{0+1}\, dy = \int_1^2 2 dy = 2$$

$$\int_C 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds = \frac{5\sqrt{5}-1}{6} + 2$$

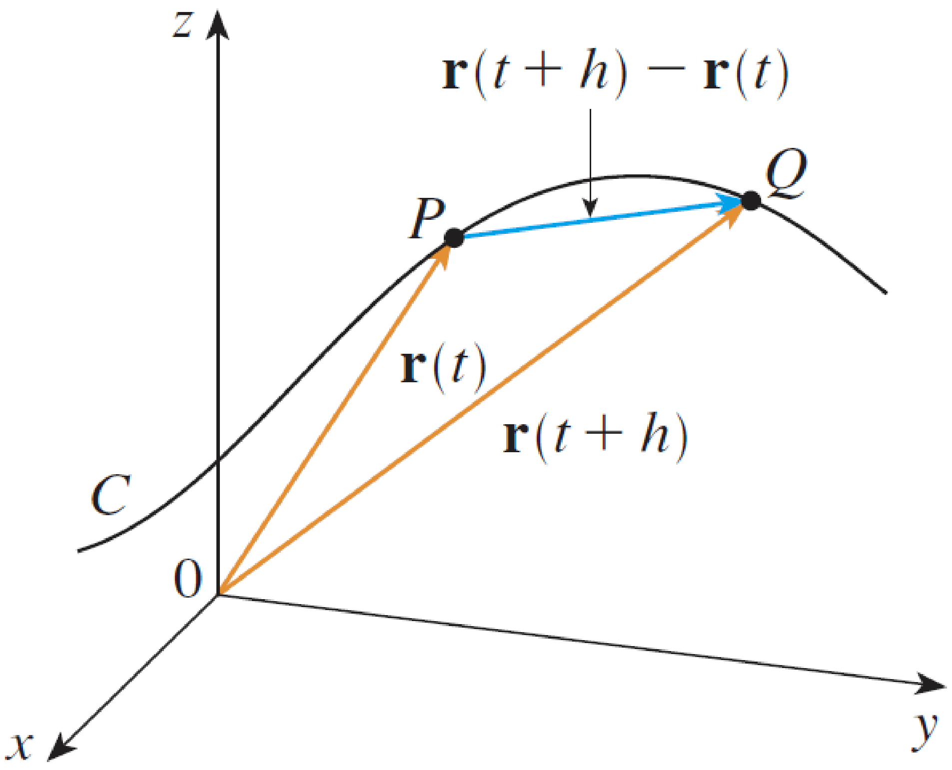
6.2 Complex Integral (複素積分)

6.2 Complex Integral (複素積分)

Q: How to represent curves in the complex plane?

6.2 Complex Integral (複素積分)

Parametrization (パラメータ表示) of Real curve (実·曲線)



Parametrization (パラメータ表示) of Complex curve (複素·曲線)

Initial Point
A: $z(a)$ i.e. $(x(a), y(a))$

Terminal Point
B: $z(b)$ i.e. $(x(b), y(b))$

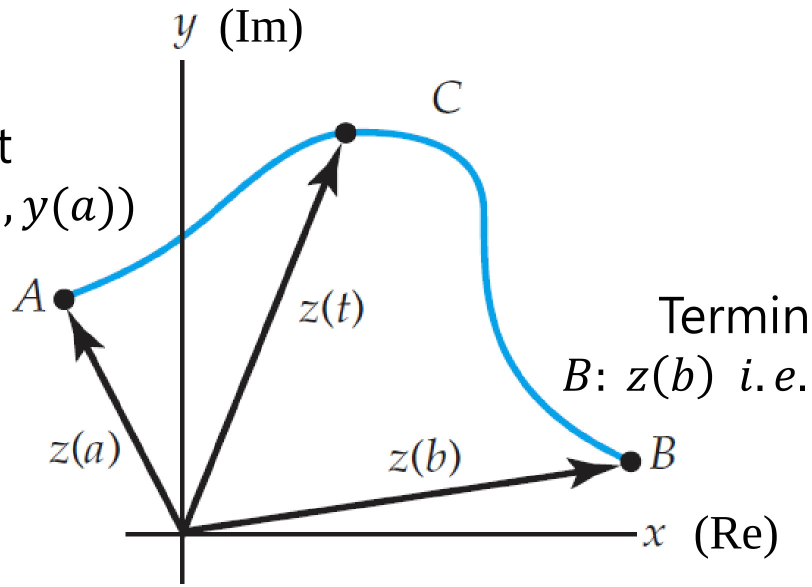
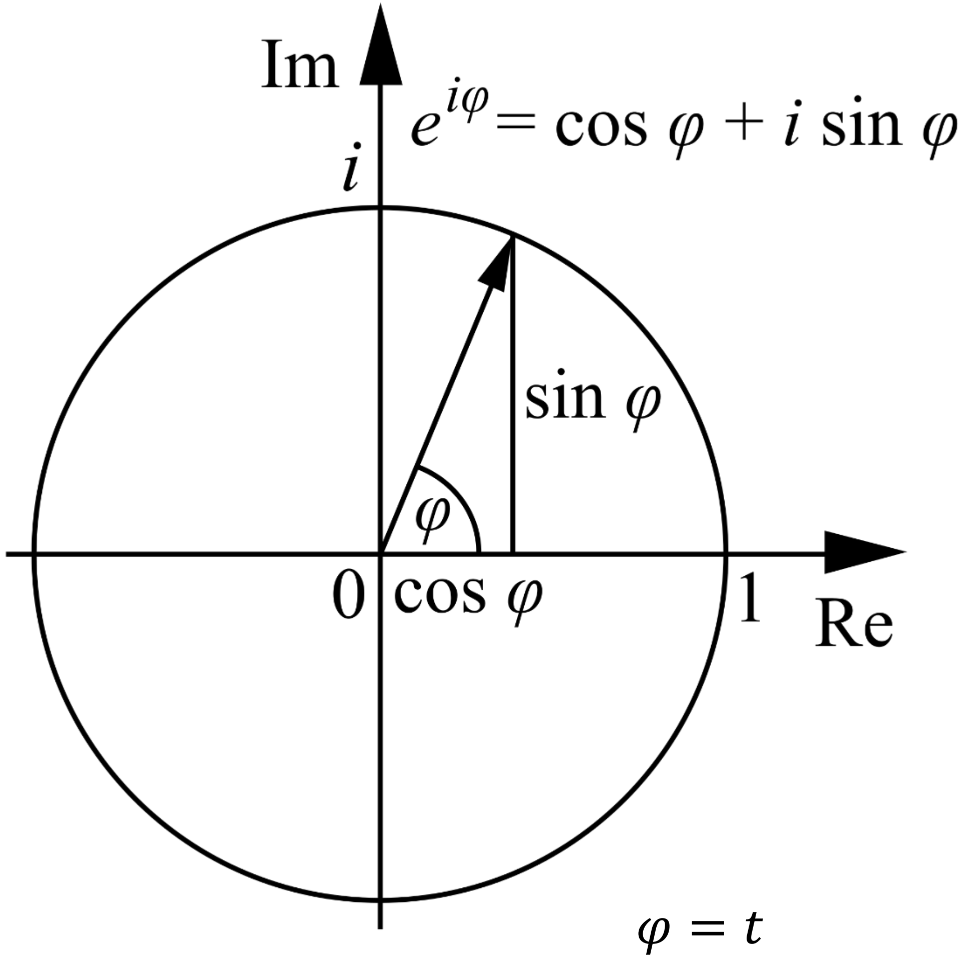


Figure 5.15 $z(t) = x(t) + iy(t)$ as a position vector

$$z(t) = x(t) + iy(t), a \leq t \leq b \quad (5.2.1)$$

The points z on the curve C is expressed by a **complex-valued** function of a **real** variable t.
This is called a parametrization of C.

6.2 Complex Integral (複素積分)



Parametrization (パラメータ表示) of Complex curve (複素·曲線)

$$z(t) = x(t) + iy(t), a \leq t \leq b$$

For example,

if $x(t) = \cos t\,, y(t) = \sin t\,, 0 \leq t \leq 2\pi$

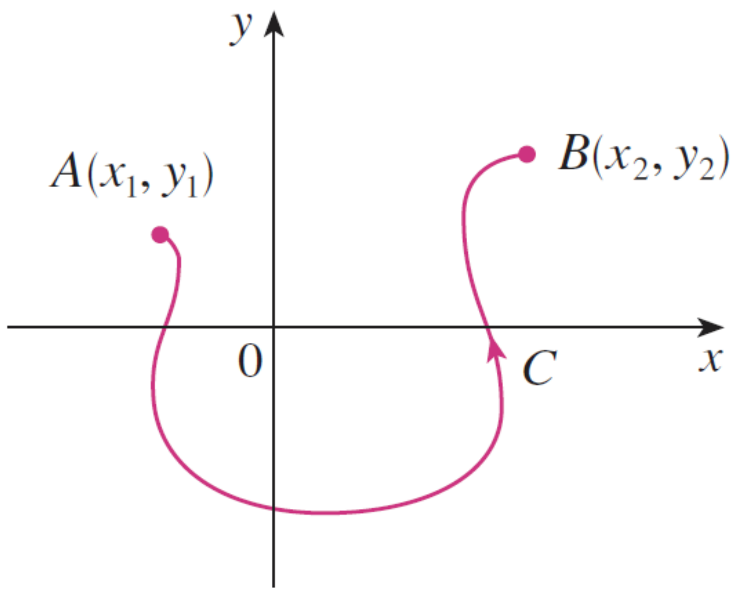
we will have

$$z(t) = \cos t + i \sin t\,, 0 \leq t \leq 2\pi$$

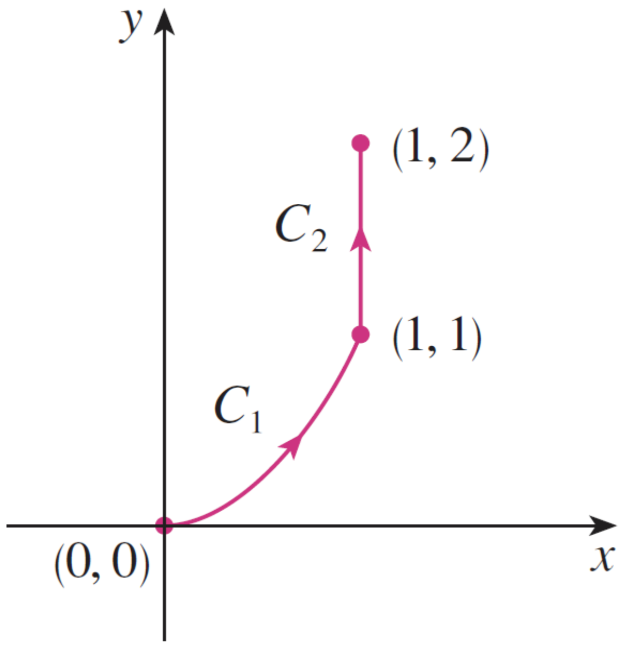
which is a parametrization of the circle C.

6.2 Complex Integral (複素積分)

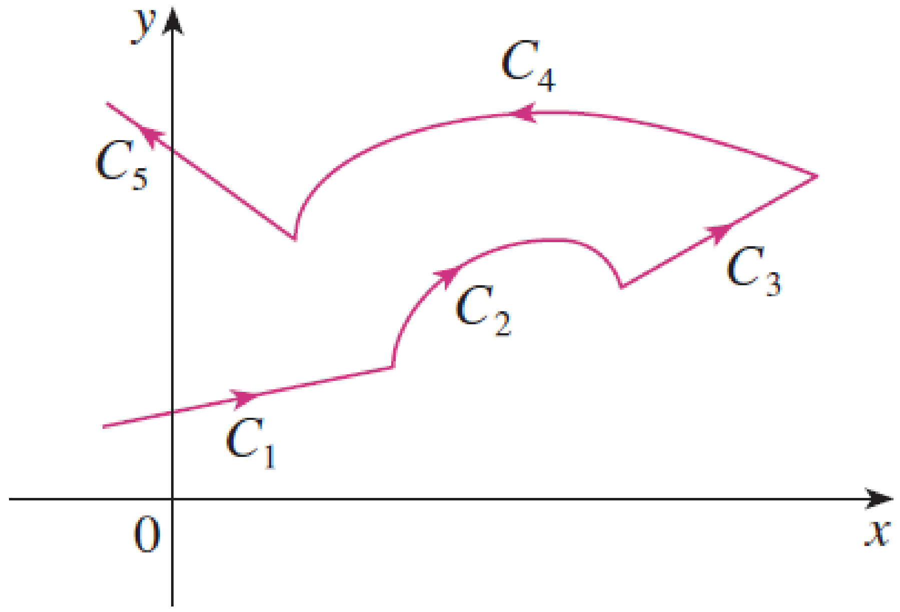
Recall Piecewise-smooth (区分的滑らか) curve



A smooth curve C



A piecewise-smooth curve $C = C_1 \cup C_2$



A piecewise-smooth curve $C = C_1 \cup C_2 \cup \cdots \cup C_5$

6.2 Complex Integral (複素積分)

Suppose the **derivative** of

$$z(t) = x(t) + iy(t), a \le \mathrm{t} \le b \quad (5.2.1)$$

is

$$z'(t) = x'(t) + iy'(t)$$

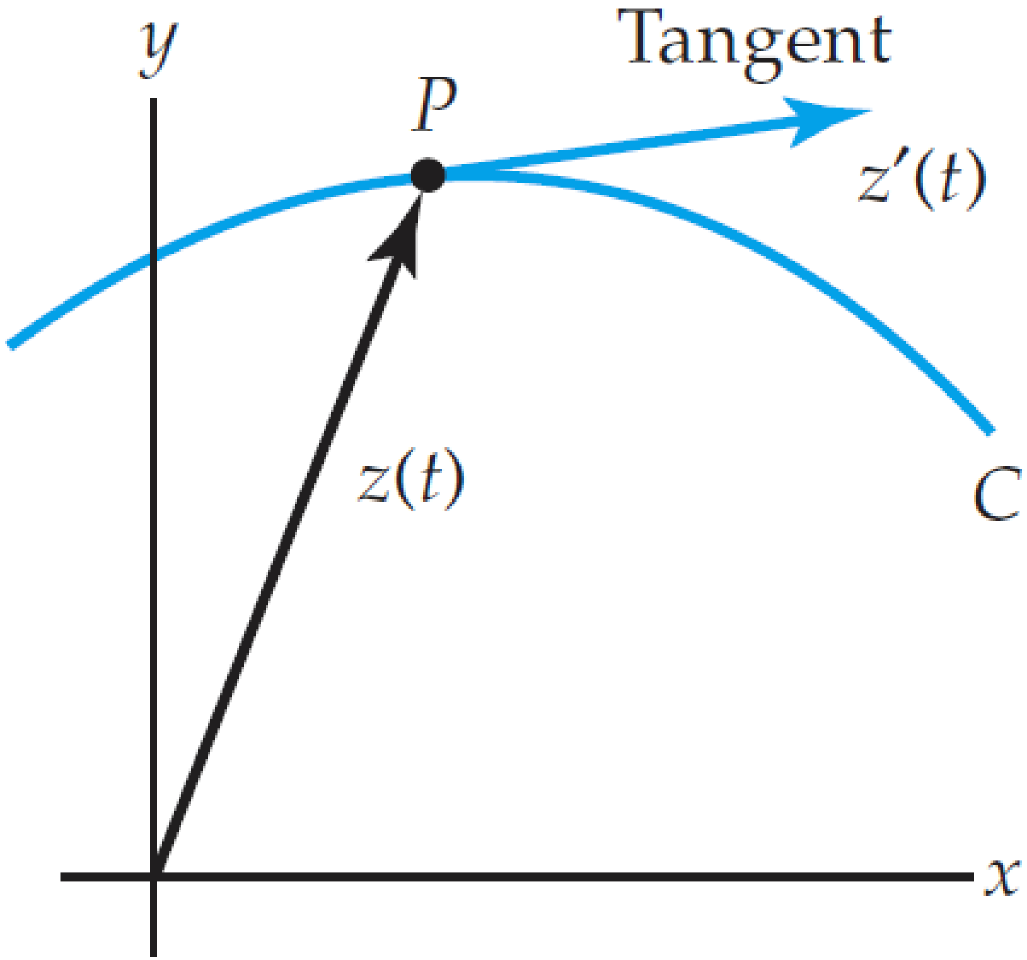


Figure 5.16 $z'(t) = x'(t) + iy'(t)$ as a tangent vector (接ベクトル)

6.2 Complex Integral (複素積分)

Smooth (滑らか) Curve

A curve C in the complex plane is called **smooth** if $z'(t)$ is **continuous** and **NEVER zero in the interval** $a \leq t \leq b$.

In other words, **a smooth curve** have **NO sharp corners or Cusps (尖点)**.

Piecewise-smooth (区分的滑らか) Curve

A piecewise-smooth curve C is **continuous EXCEPT possibly at the points** where the **component smooth curves** $C_1, C_2, \ldots, C_n$ **are joined together**.

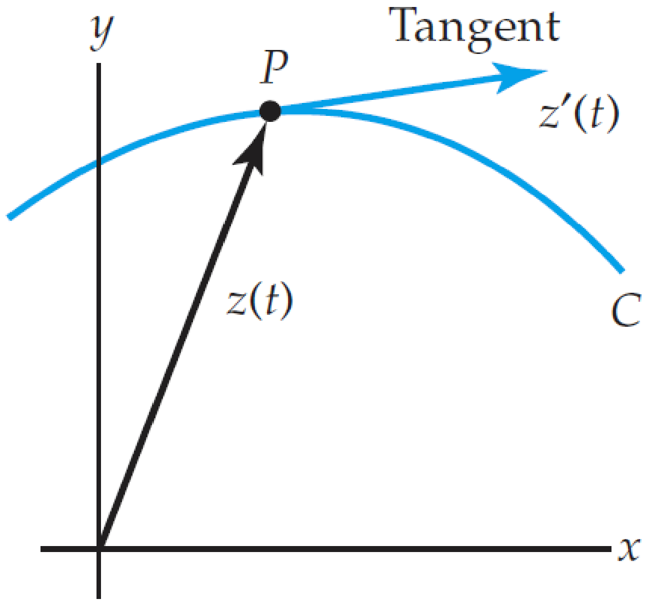


Figure 5.16 $z'(t) = x'(t) + iy'(t)$ as a tangent vector (接ベクトル)

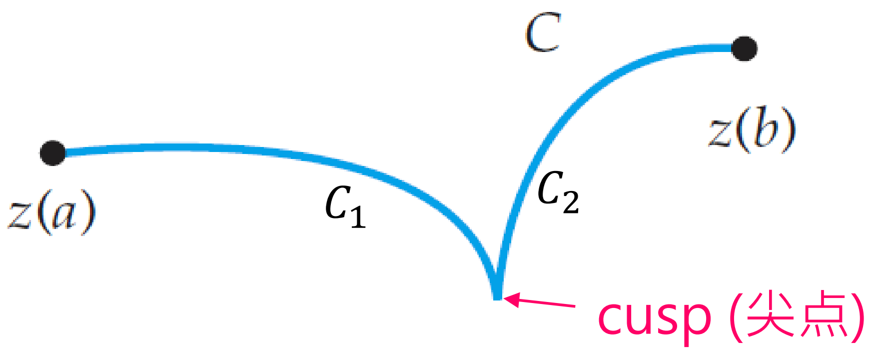
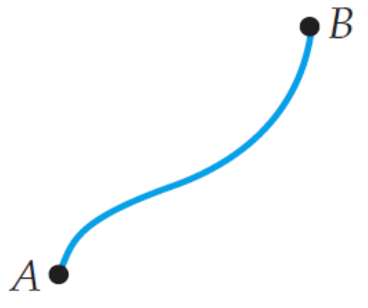


Figure 5.17 Curve C is not smooth because it has a cusp

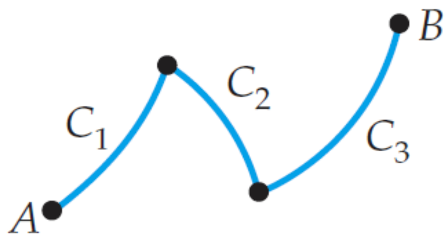
6.2 Complex Integral (複素積分)

Simple (単一) Curve

A curve C in the **complex plane** is said to be a **simple** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for initial point $t = a$ and terminal point $t = b$.



Smooth, simple, not closed

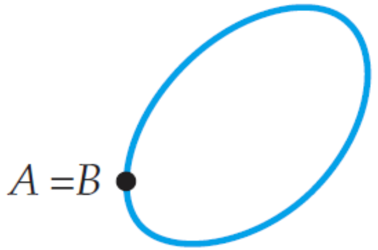


Piecewise-smooth, simple, not closed

Closed (閉) Curve

C is a **closed curve** if $z(a) = z(b)$.

Smooth, closed, not simple



Smooth, simple, closed

Simple Closed Curve (単一閉曲線)

C is a **simple closed curve** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$.

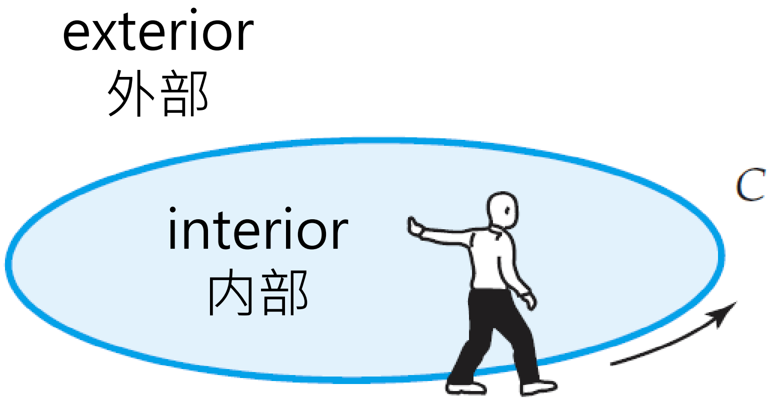
6.2 Complex Integral (複素積分)

Contour

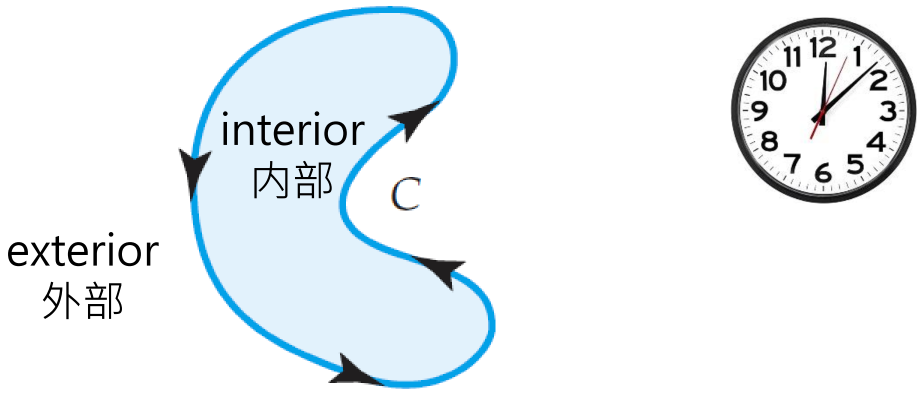
A piecewise-smooth curve C is called a **Contour** or Path.

Direction (向き) on a Contour

- We define the **positive direction** on a contour C corresponding to increasing values of the parameter t.
- Roughly, for a simple closed curve C, the **positive direction** is the **counterclockwise (左回りの) direction** or the direction that **a person must walk on C and keep the interior (内部) of C at the left hand**.
- The **negative direction** on a contour C is the direction **opposite (反対の) the positive direction**.
- **Notice**: If C has positive direction, **then** its opposite curve can be denoted by $-C$.



(a) Positive direction



(b) Positive direction

Figure 5.18 Interior of each curve is at the left hand

Note: Find more explanations in Page 247 ~ 250 of the textbook.

Definition 5.3 Complex Integral (複素積分)

An integral of a function $f(z)$ defined by

$$\int_C f(z)\,dz = \lim_{\|\Delta z_{max}\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \qquad (5.2.2)$$

is called a **complex integral**, where z is a complex number, and $f(z)$ is defined on a contour C (積分路). (Here the norm $\|\Delta z_{max}\|$ defines the length of the longest subinterval (部分区間))

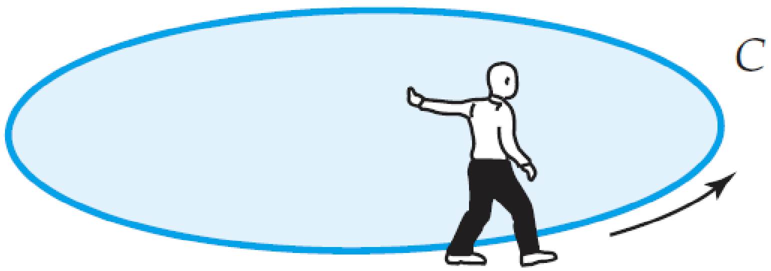
- If the limit in (5.2.2) exists, then $f(z)$ is said to be **integrable (可積分の)** on C.
- The **limit exists** whenever **if** $f(z)$ is **continuous at all points on** C, where C is either **smooth or piecewise-smooth**.
- The **Complex Integral** $\int_C f(z)dz$ has a more common name: **Contour Integral**.

6.2 Complex Integral (複素積分)

- Specially, we will use the notation

$$\oint_C f(z)dz$$

as a complex integral around a positively oriented closed curve C.



Closed curve C with Positive direction

6.2 Complex Integral (複素積分)

Integral for Complex-Valued Function of a Real Variable

Example

If t represents a **real** variable, then the output of the function $f(t) = (2t + i)^2$ is a **complex** number. For $t = 2$,

$$f(2) = (2 \cdot 2 + i)^2 = 16 + 8i + i^2 = 15 + 8i.$$

In general, if f_1 and f_2 are **real-valued** functions of a **real** variable t (that is, **real** functions), then $f(t) = f_1(t) + if_2(t)$ is a **complex-valued** function of a **real** variable t.

When we consider the interval $0 \le t \le 1$,

$$\int_0^1 \underbrace{(2t + i)^2}_{f(t)} dt = \int_0^1 (4t^2 - 1 + i4t)\, dt = \int_0^1 \underbrace{(4t^2 - 1)}_{f_1(t)} dt + i \int_0^1 \underbrace{4t}_{f_2(t)} dt = \left(\frac{4}{3} \cdot t^3 - t\right)\Big|_0^1 + i \cdot 2t^2\Big|_0^1 = \frac{1}{3} + 2i$$

Then we can define **the integral (積分) of the complex-valued function** $f(t) = f_1(t) + if_2(t)$ **on interval** $a \le t \le b$ **as**

$$\int_a^b f(t)\, dt = \int_a^b f_1(t)\, dt + i \int_a^b f_2(t)\, dt \qquad (5.2.4)$$

Theorem 5.1 How to Compute a Contour Integral (i.e. Complex Integral) of a Complex Variable

If f is **continuous** on a **smooth curve** C given by the parametrization (パラメータ表示) $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt \qquad (5.2.11)$$

where

$$f\big(z(t)\big)z'(t) = f\big(z(t)\big)[x'(t) + iy'(t)] = \Big[u\big(x(t), y(t)\big) + iv\big(x(t), y(t)\big)\Big][x'(t) + iy'(t)]$$

EXAMPLE (例題) 5.2.1 **Evaluating a Contour Integral**

Evaluate $\int_C \bar{z}\, dz$, where C is given by $z(t) = x(t) + iy(t)$, $x(t) = 3t$, $y(t) = t^2$, $-1 \le t \le 4$.

Solution (解答):

$\because \quad z(t) = x(t) + iy(t) = 3t + it^2$

$\therefore \quad f\big(z(t)\big) = \overline{z(t)} = x(t) - iy(t) = 3t - it^2$ and $z'(t) = x'(t) + iy'(t) = 3 + i2t$

Then by Equation (5.2.11)

$$\int_C \bar{z}\, dz = \int_{-1}^{4} f\big(z(t)\big) z'(t) dt = \int_{-1}^{4} (3t - it^2)(3 + i2t) dt = \int_{-1}^{4} (2t^3 + 9t + 3t^2 i) dt$$

By using Equation (5.2.4) in this Lecture, we have

$$\int_C \bar{z}\, dz = \int_{-1}^{4} (2t^3 + 9t) dt + i \int_{-1}^{4} 3t^2 dt = \left(2 \cdot \frac{1}{4} \cdot t^4 + 9 \cdot \frac{1}{2} \cdot t^2\right)\Bigg|_{-1}^{4} + i \cdot t^3\Big|_{-1}^{4} = 195 + 65i$$

EXAMPLE (例題) 5.2.2 **Evaluating a Contour Integral**

Evaluate $\oint_C \frac{1}{z} dz$, where C is the circle $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$.

Solution (解答):

$\because \quad z(t) = \cos t + i \sin t = e^{it}$

$\therefore \quad f(z(t)) = \frac{1}{z(t)} = \frac{1}{e^{it}} = e^{-it}$ and $z'(t) = ie^{it}$

Then by Equation (5.2.11)

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} f(z(t))z'(t)dt = \int_0^{2\pi} (e^{-it})ie^{it} dt$$

$$= i\int_0^{2\pi} e^{-it+it} dt = i\int_0^{2\pi} e^0 dt = i\int_0^{2\pi} dt = i \cdot t\Big|_0^{2\pi} = 2\pi i$$

Theorem 5.2 Properties of Contour Integrals

Suppose **the functions** f and g are **continuous in a domain** D, and C is a **smooth curve lying entirely in** D. Then

(i) $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, where k is a complex constant.

(ii) $\displaystyle\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation (向き) of C.

EXAMPLE (例題) 5.2.3 C is a Piecewise-Smooth Curve

Evaluate $\int_C (x^2 + iy^2)\, dz$, where C is the contour shown in Figure 5.20.

Solution (解答):

$f(z) = x^2 + iy^2$ From Theorem 5.2(iii), we have

$$\int_C (x^2 + iy^2)\, dz = \int_{C_1} (x^2 + iy^2)\, dz + \int_{C_2} (x^2 + iy^2)\, dz$$

From the Figure 5.20, we know the curves

① C_1 is $y = x$, when $0 \le x < 1$

Therefore, $(x(t), y(t))$ becomes $(x(t), x(t))$, it makes sense that we can directly use x as parameter t, then

$$z(x) = x + ix \qquad z'(x) = 1 + i$$

$$f\big(z(x)\big) = x^2 + ix^2$$

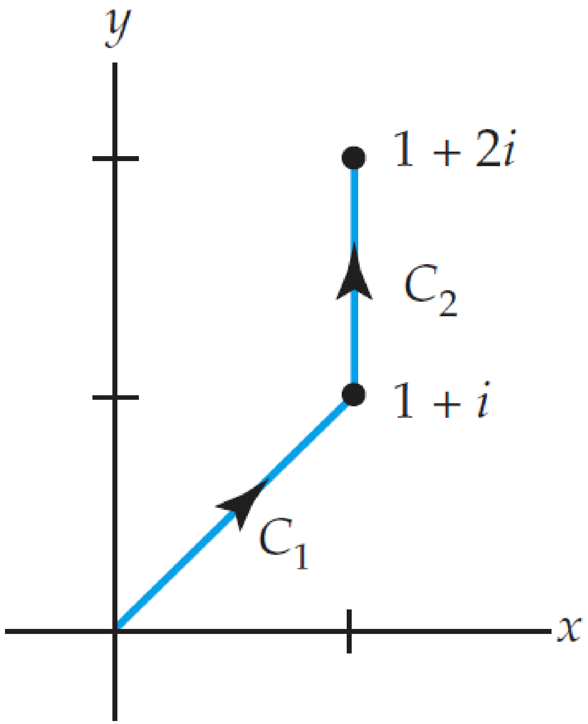


Figure 5.20 Contour $C = C_1 \cup C_2$ is piecewise-smooth

6.2 Complex Integral (複素積分)

Solution (解答)(cont.):

Then by Equation (5.2.11)

$$\int_{C_1}(x^2+iy^2)\,dz=\int_0^1 f\big(z(x)\big)z'(x)\,dx=\int_0^1(x^2+ix^2)(1+i)\,dx$$

$$=(1+i)^2\int_0^1 x^2\,dx=(1+i)^2\cdot\frac{1}{3}\cdot x^3\Big|_0^1=\frac{2}{3}i$$

② C_2 is $x=1$, when $1\le y\le 2$

Therefore, $(x(t),y(t))$ becomes $\big(1,y(t)\big)$, it makes sense that we can directly use y as parameter t, then

$z(y)=1+iy\quad z'(y)=0+i=i\qquad f\big(z(y)\big)=1+iy^2$

Then by Equation (5.2.11)

$$\int_{C_2}(x^2+iy^2)\,dz=\int_1^2 f\big(z(y)\big)z'(y)\,dy=\int_1^2(1+iy^2)i\,dy=-\int_1^2 y^2\,dy+i\int_1^2 1\,dy=-\frac{7}{3}+i$$

by Equation (5.2.4)

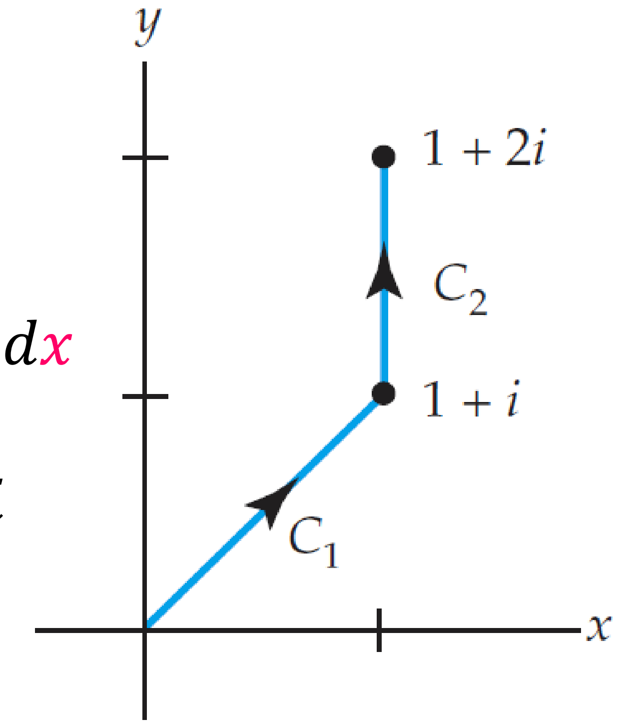


Figure 5.20 Contour $C=C_1\cup C_2$ is piecewise-smooth

Combining the results of ① and ②, we have $\int_C (x^2+iy^2)\,dz=\frac{2}{3}i+\left(-\frac{7}{3}+\mathrm{i}\right)=-\frac{7}{3}+\frac{5}{3}i$

Review for Lecture 6

- Real Line Integral
- Smooth / Piecewise-Smooth Curve
- Simple, Closed Curve
- Complex Integral (Contour Integral)
- How to Compute Complex Integral

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

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[2] Calculus, 6th Edition, James Stewart, Thomas Brooks/Cole, 2009

[3] Wikipedia