Lecture 7

7.1 Cauchy's Integral Theorem (コーシーの積分定理)

7.1.1 Simply and Multiply Connected Domains

7.1.2 Cauchy's Integral Theorem for Simply Connected Domains

7.1.3 Cauchy's Integral Theorem for Multiply Connected Domains

7.1 Cauchy's Integral Theorem

コーシーの積分定理

(i.e. Cauchy-Goursat Theorem)

In this subsection, we shall concentrate on **contour integrals**, where the **contour** C is a **simple closed curve** with **a positive (counterclockwise) orientation**.

7.1.1 Simply Connected (単連結) Domains

and

Multiply Connected (多重連結) Domains

7.1.1 Simply and Multiply Connected Domains

Recall the **polar form of complex plane** in the Lecture 1.

1.4 Polar form (極形式) of Complex Plane (複素平面)

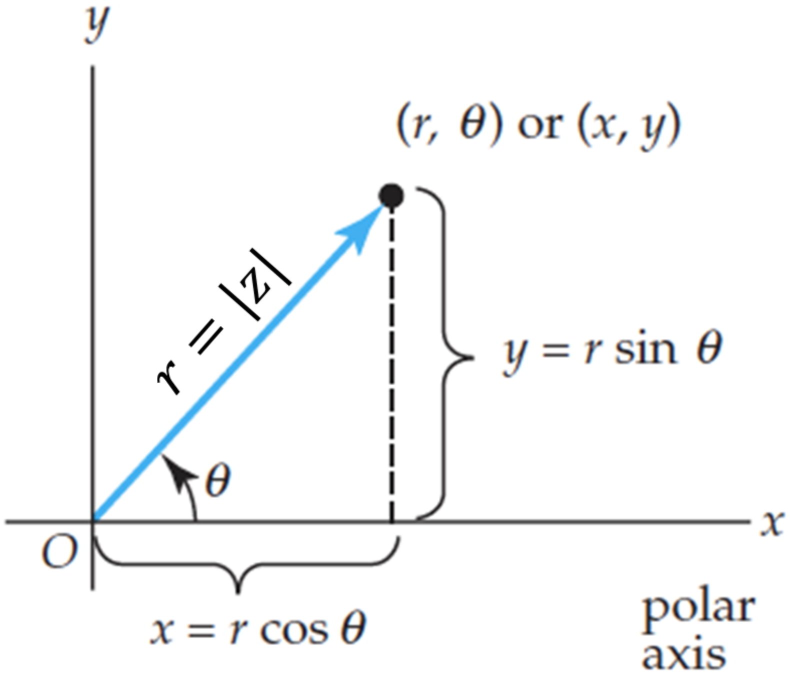


Figure 1.7 **Polar coordinates** (極座標系) in the complex plane

$$z = x + iy$$
$$= (r\cos\theta) + i(r\sin\theta)$$
$$= r(\cos\theta + i\sin\theta) \qquad (1.3.1)$$

$$z = re^{i\theta}$$

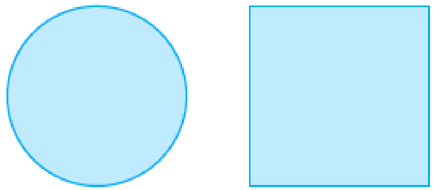
By using Euler's Formula
$e^{i\theta} = \cos\theta + i\sin\theta$

We say that (1.3.1) is the **polar form** (極形式) of the complex number z.

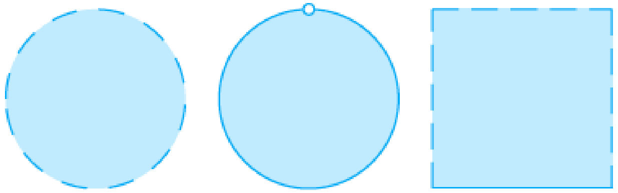
We call $r = |z| = \sqrt{x^2 + y^2}$ as **modulus** or **magnitude (複素数の絶対値) of** z,

$\theta = \arg(z) = \arctan\frac{y}{x}$ as **argument (偏角) of** z. (Notice quadrant)

7.1.1 Simply and Multiply Connected Domains



(a) Closed sets



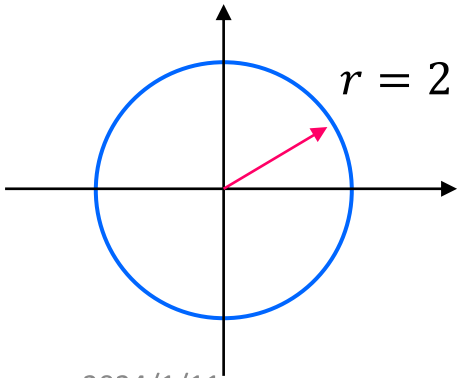
(b) Sets that are not closed

A **closed set (閉集合)** contains all its boundary points.

A **bounded set (有界集合)** is contained within some disk.

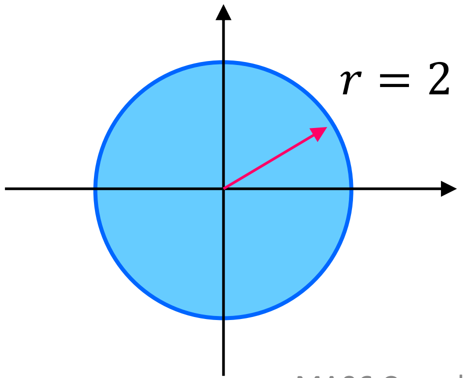
circle
(円)

$|z| = 2$



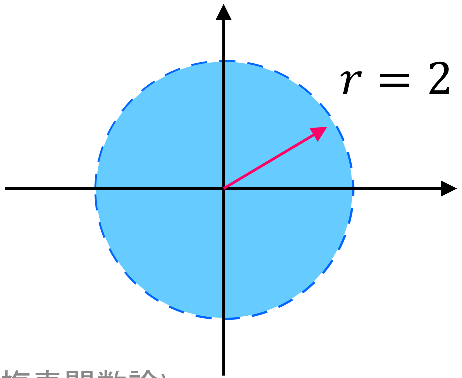
closed disk
(閉円板)

$|z| \leq 2$



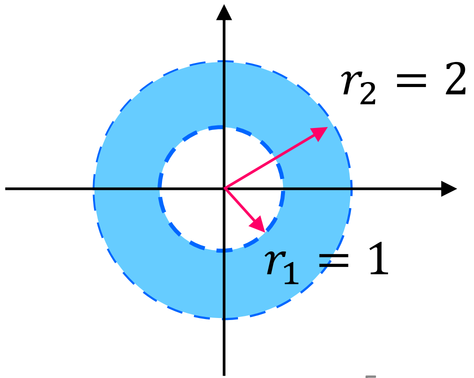
open disk
(開円板)

$|z| < 2$



open circular
annulus (開円環)

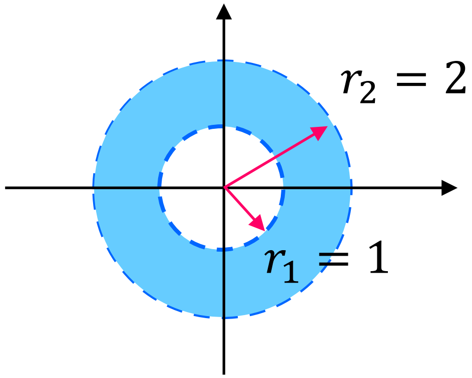
$1 < |z| < 2$



7.1.1 Simply and Multiply Connected Domains

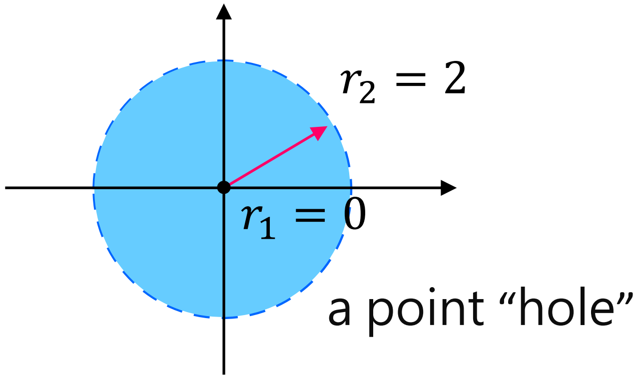
open circular annulus (開円環)

$1 < |z| < 2$



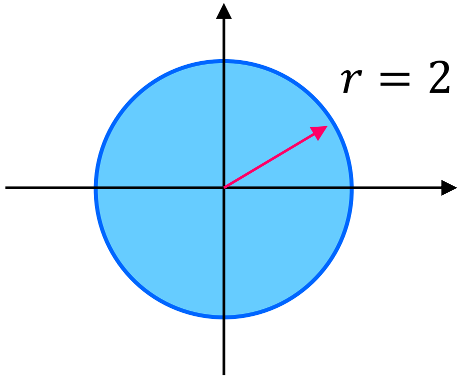
open circular annulus (開円環)

$0 < |z| < 2$



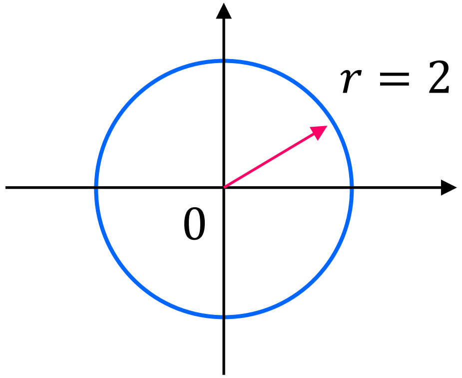
closed disk (閉円板)

$|z| \leq 2$



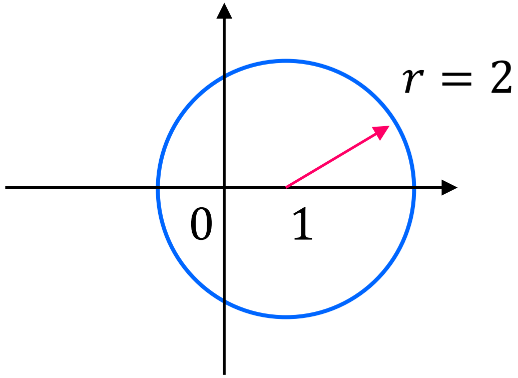
circle (円)

$|z| = 2$



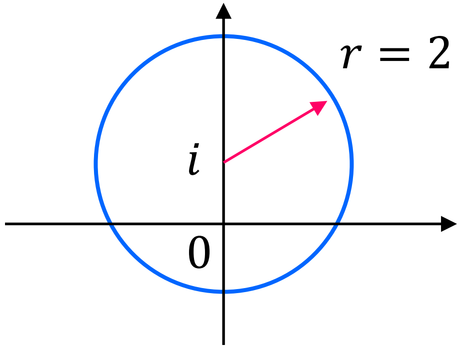
circle (円)

$|z - 1| = 2$



circle (円)

$|z - i| = 2$



circle (円)

$|z - z_0| = r$

7.1.1 Simply and Multiply Connected Domains

Simply Connected (単連結) Domains

We say that **a domain** D **is simply connected if every simple closed contour** C **lying entirely in** D **can be shrunk to a point** (ポイントに縮小する) **without leaving** D. (See Figure 5.26.)

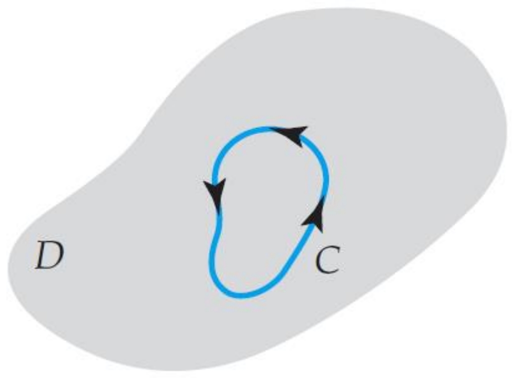


Figure 5.26 Simply connected domain D

In other words, a simply connected domain **has no "holes" in** it.

Multiply Connected (多重連結) Domains

A domain that is not simply connected is called **a multiply connected domain**. (See Figure 5.27.)

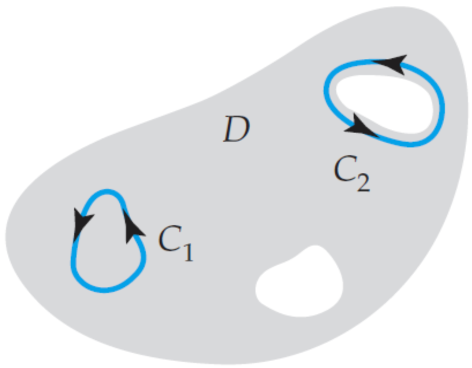


Figure 5.27 Multiply connected domain D

In other words, a multiply connected domain **has "holes" in it.**

For example, (1)the open disk (開円板) defined by $|z| < 2$ is a simply connected domain; (2)the open circular annulus (開円環) defined by $1 < |z| < 2$ is a doubly (i.e. multiply) connected domain.

7.1.2 Cauchy's Integral Theorem

(コーシーの積分定理)

for Simply Connected Domains

Theorem 5.4 Cauchy's Integral Theorem コーシーの積分定理 (i.e. Cauchy-Goursat Theorem)

Suppose that a function f is **analytic (解析的)** in a **simply connected (単連結) domain** D. Then for every simple closed contour C in D, we have

$$\oint_C f(z)dz = 0$$

Because the **interior (内部) of a simple closed contour** is a **simply connected domain**, the **Theorem 5.4** can be **rewritten** in the slightly more practical manner:

If f is **analytic at all points within and on** a **simple closed contour** C, then

$$\oint_C f(z)dz = 0 \qquad (5.3.4)$$

EXAMPLE (例題) 5.3.1 Applying the Cauchy's Integral Theorem

Evaluate $\oint_C e^z dz$, where the contour C is shown in Figure 5.28.

Solution (解答):

The function $f(z) = e^z$ is entire (整函数) and consequently is analytic at all points within and on the simple closed contour C.

Then from the Cauchy's Integral Theorem given in (5.3.4) that

$$\oint_C e^z dz = 0$$

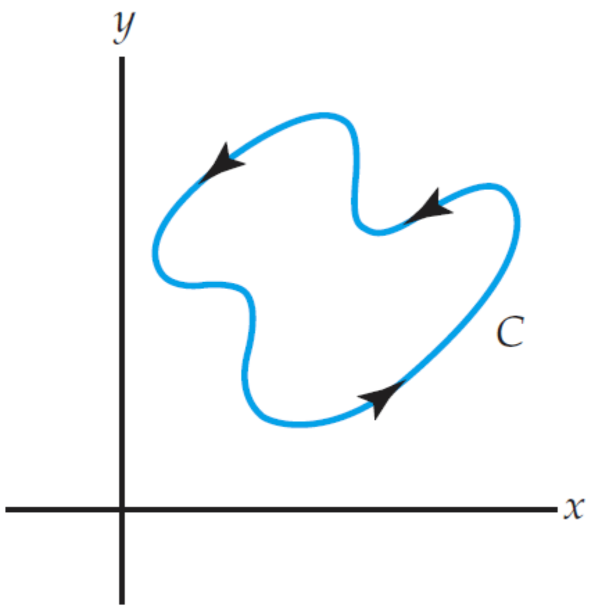


Figure 5.28 Contour for Example 5.3.1

Indeed, from Example 5.3.1, it follows that **for any simple closed contour C and any entire** function (整函数) f, such as

$$f(z) = \sin z,$$

$$f(z) = \cos z,$$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,\ \ n = 0, 1, 2, \ldots$$

we have

$$\oint_C \sin z \, dz = 0,$$

$$\oint_C \cos z \, dz = 0,$$

$$\oint_C p(z) dz = 0$$

and so on.

EXAMPLE (例題) 5.3.2 Applying the Cauchy's Integral Theorem

Evaluate $\oint_C \frac{1}{z^2} dz$, where the contour C is the ellipse (楕円) $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$.

Solution (解答):

The rational function $\frac{1}{z^2}$ is analytic everywhere except at $z = 0$.

But $z = 0$ is not a point interior to or on the simple closed elliptical contour C.

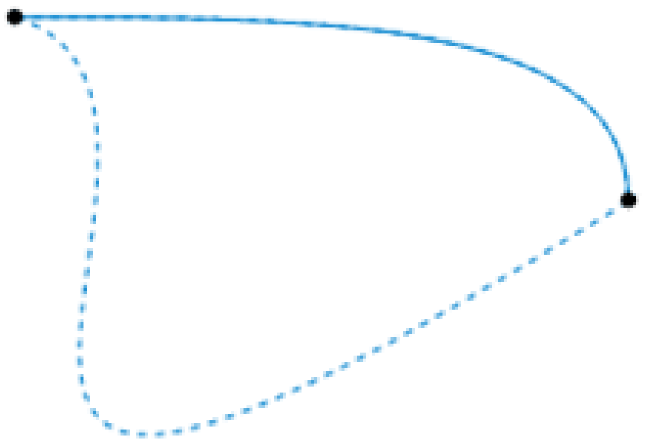
Thus, from the Cauchy's Integral Theorem given in (5.3.4) we have that

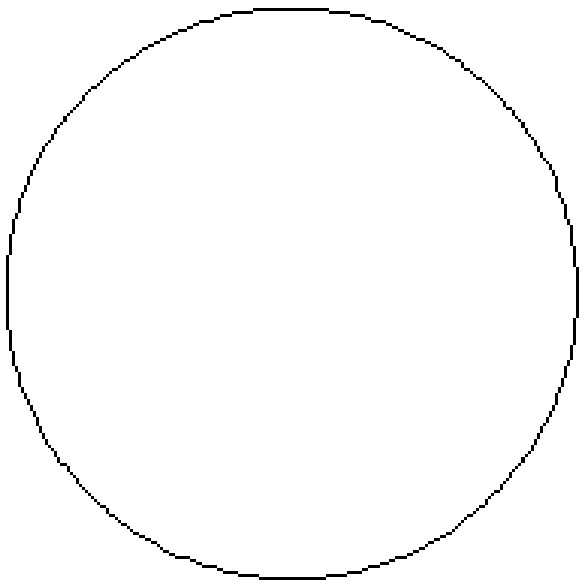
$$\oint_C \frac{1}{z^2} dz = 0$$

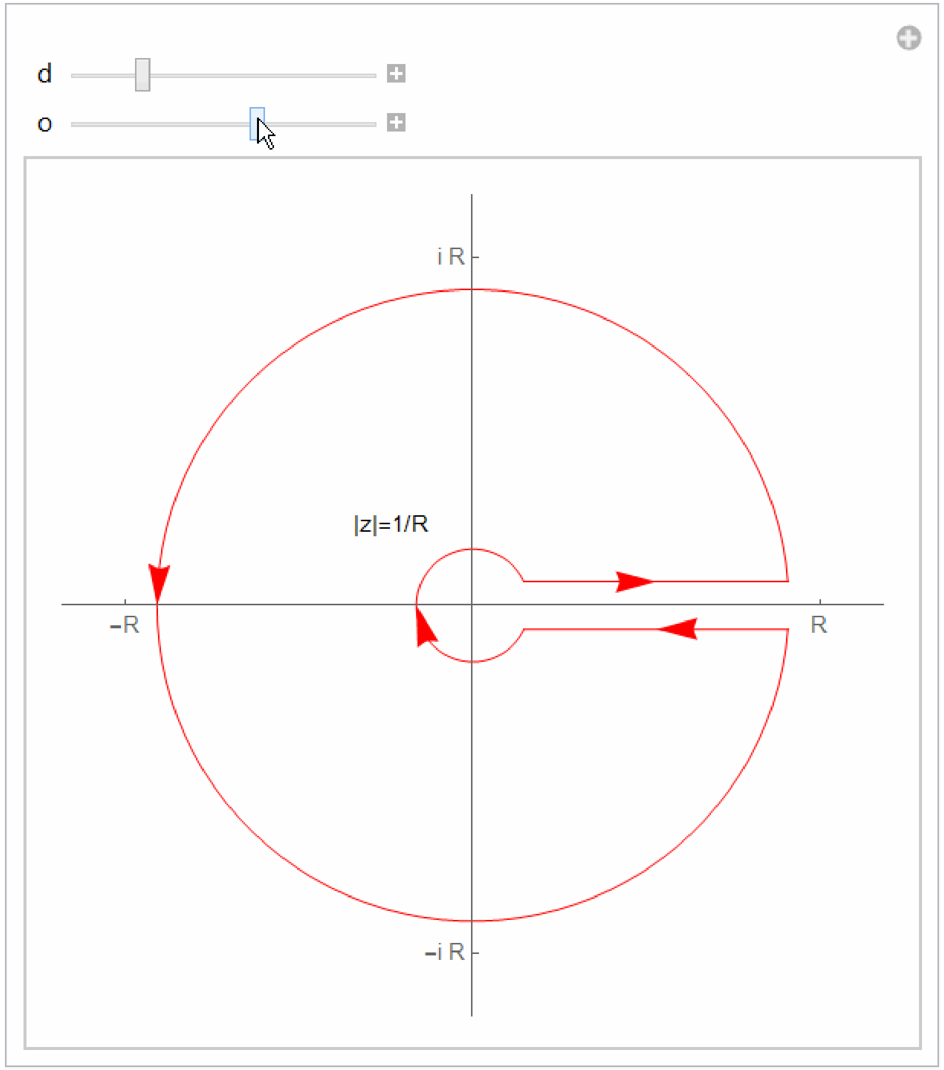
7.1.3 Cauchy’s Integral Theorem

for Multiply Connected Domains

7.1.3 Cauchy’s Integral Theorem for Multiply Connected Domains







Continuous deformation (連続変形) of a contour

7.1.3 Cauchy's Integral Theorem for Multiply Connected Domains

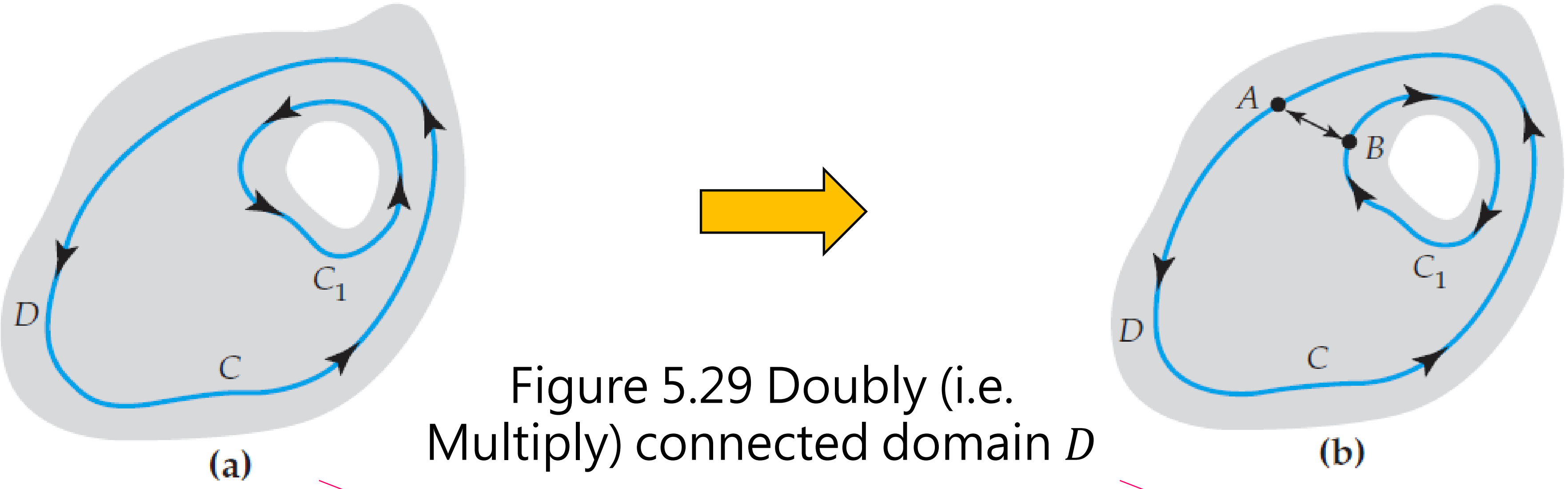


Figure 5.29 Doubly (i.e. Multiply) connected domain D

$$\oint_C f(z)dz + \int_{AB} f(z)dz + \oint_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$$

$$\Rightarrow \oint_C f(z)dz + \oint_{-C_1} f(z)dz = 0 \quad \Rightarrow \oint_C f(z)dz = \oint_{C_1} f(z)dz \qquad (5.3.5)$$

The above result is sometimes called the <u>principle of deformation (変形) of contours</u> because we can think of **the contour C_1 as a continuous deformation (連続変形) of the contour C**.

In other words, (5.3.5) allows us to **evaluate an integral (積分) over a complicated (複雑な) simple closed contour C** **<u>by replacing</u> C with a contour C_1 that is more convenient (便利な)**.

7.1.3 Cauchy's Integral Theorem for Multiply Connected Domains

Additional Knowledge: **Common Parametric Curves in the Complex Plane**

Line

A parametrization of the line containing the points z_0 and z_1 is:

$$z(t) = z_0(1-t) + z_1 t, \qquad -\infty \leq t \leq \infty. \qquad (2.2.7)$$

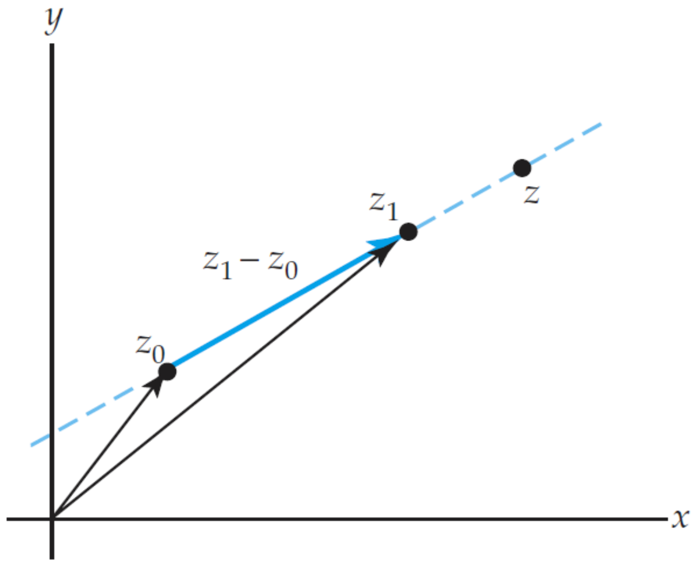


Figure 2.4 Parametrization of a line

Circle

A parametrization of the circle centered at z_0 with radius r is:

$$z(t) = z_0 + r(\cos t + i \sin t), \qquad 0 \leq t \leq 2\pi. \quad (2.2.9)$$

In exponential notation, this parametrization is:

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi. \qquad (2.2.10)$$

EXAMPLE (例題) 5.3.3 Applying Deformation of Contours

Evaluate $\oint_C \frac{1}{z-i} dz$, where the contour C is shown in black color in Figure 5.30. (Notice that there is a point "hole" at $(0,1)$.)

Solution (解答):

From (5.3.5), we choose the more convenient circular contour C_1 drawn in blue color in the Figure 5.30.

By taking the radius (半径) of the circle to be $r = 1$, we are guaranteed (保証される) that C_1 lies within C. In other words, C_1 is the circle $|z - i| = 1$, which from (2.2.10) can be parametrized by $z = i + e^{it}$, $0 \le t \le 2\pi$.

Thus $z - i = e^{it}$ and $dz = ie^{it}dt$, we obtain

$$\oint_C \frac{1}{z-i} dz = \oint_{C_1} \frac{1}{z-i} dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

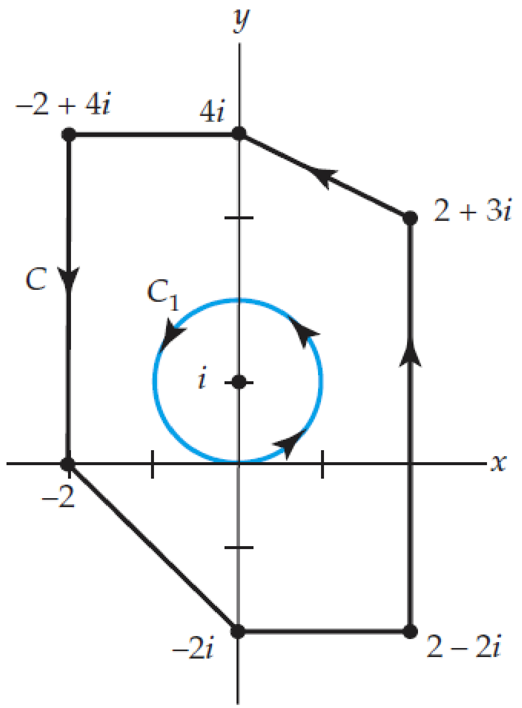


Figure 5.30 We use the simpler contour C_1 in Example 5.3.3.

7.1.3 Cauchy’s Integral Theorem for Multiply Connected Domains

The result obtained in Example 5.3.3 can be generalized.

By using the principle of deformation of contours (5.3.5), it can be shown that if z_0 is **any constant complex number interior to any simple closed contour** C, then **for an integer** n we have

$$\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases} \qquad (5.3.6)$$

EXAMPLE (例題) 5.3.4 Applying Formula (5.3.6)

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where the contour C is the circle $|z-2|=2$.

Solution (解答):

Because the denominator factors as $z^2+2z-3=(z-1)(z+3)$ the integrand fails to be analytic at $z=1$ and $z=-3$. Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. Now by partial fractions

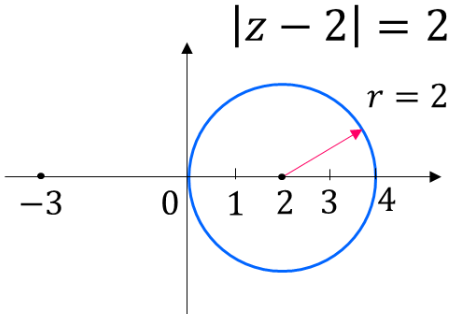
$$\frac{5z+7}{z^2+2z-3}=\frac{5z+7}{(z-1)(z+3)}=\frac{3(z+3)}{(z-1)(z+3)}+\frac{2(z-1)}{(z-1)(z+3)}=\frac{3}{z-1}+\frac{2}{z+3}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=\oint_C \frac{3}{z-1}dz+\oint_C \frac{2}{z+3}dz=3\oint_C \frac{1}{z-1}dz+2\oint_C \frac{1}{z+3}dz \qquad (5.3.7)$$

By (5.3.6), the first integral in (5.3.7) has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy's Integral Theorem.

Hence, (5.3.7) becomes

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=3\cdot(2\pi i)+2\cdot(0)=6\pi i$$



Theorem 5.5 Cauchy's Integral Theorem for Multiply Connected Domains

Suppose $C,\ C_1,\ldots,C_n$ are **simple closed curves with a positive orientation** such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f **is analytic on each contour and at each point interior to** C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz \quad (5.3.8)$$

EXAMPLE (例題) 5.3.5 Applying Theorem 5.5

Evaluate $\oint_C \frac{1}{z^2+1} dz$, where the contour C is the circle $|z| = 3$.

Solution (解答): We know that $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$,

Consequently, the integrand $1/(z^2+1)$ is not analytic at $z = i$ and at $z = -i$. Both of these points lie within the contour C.

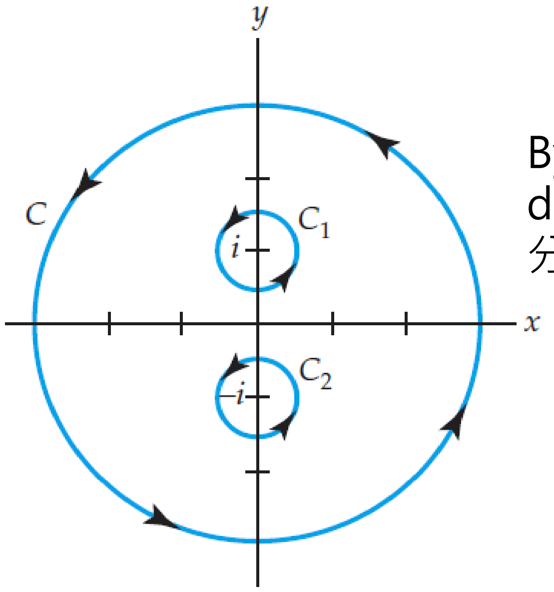


Figure 5.32 Contour for Example 5.3.5

By using partial fraction decomposition (部分分数分解):

$$\frac{1}{(z+i)(z-i)} = \frac{\frac{1}{2i}(z+i)}{(z+i)(z-i)} - \frac{\frac{1}{2i}(z-i)}{(z+i)(z-i)} = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}$$

$$\oint_C \frac{1}{z^2+1} dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) dz$$

We now choose to surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. Specifically, the choice $|z - i| = \frac{1}{2}$ for C_1 and $|z + i| = \frac{1}{2}$ for C_2 will suffice (十分である). See Figure 5.32. From Theorem 5.5 we can write:

Solution (解答)(cont.):

$$\oint_C \frac{1}{z^2+1}dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)dz = \oint_{C_1} \frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)dz + \oint_{C_2} \frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)dz$$

$$= \frac{1}{2i}\oint_{C_1}\frac{1}{z-i}dz - \underbrace{\frac{1}{2i}\oint_{C_1}\frac{1}{z+i}dz}_{0} + \underbrace{\frac{1}{2i}\oint_{C_2}\frac{1}{z-i}dz}_{0} - \frac{1}{2i}\oint_{C_2}\frac{1}{z+i}dz \qquad (5.3.9)$$

Because $1/(z+i)$ is analytic on C_1 and at each point in its interior and because $1/(z-i)$ is analytic on C_2 and at each point in its interior, it follows from (5.3.4) that the second and third integrals in (5.3.9) are zero.

$$\oint_C \frac{1}{z^2+1}dz = \frac{1}{2i}\oint_{C_1}\frac{1}{z-i}dz - 0 + 0 - \frac{1}{2i}\oint_{C_2}\frac{1}{z+i}dz$$

Moreover, it follows from (5.3.6), with $n = 1$, that $\oint_{C_1}\frac{1}{z-i}dz = 2\pi i$ and $\oint_{C_2}\frac{1}{z+i}dz = 2\pi i$

Then $$\oint_C \frac{1}{z^2+1}dz = \frac{1}{2i}\oint_{C_1}\frac{1}{z-i}dz - \frac{1}{2i}\oint_{C_2}\frac{1}{z+i}dz = \frac{2\pi i}{2i} - \frac{2\pi i}{2i} = 0$$

Review for Lecture 7

- Simply and Multiply Connected Domains
- Cauchy's Integral Theorem for Simply Connected Domains
- Cauchy's Integral Theorem for Multiply Connected Domains

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia