Lecture 9

*9.1 Review of (Real) Sequences and Series

9.2 (Complex) Sequences and Series (複数)数列と級数

9.3 Series Tests

Cauchy's integral formula for derivatives (Theorem 5.10) indicates that if a function f is analytic at a point z_0, **then it possesses derivatives of all orders at that point**.

As a consequence of this result we shall see that f can always be expanded in **a power series** centered at that point.

On the other hand, if f fails to be analytic at z_0, we may still be able to expand it in a different kind of series known as a **Laurent series**.

*9.1 Review of (Real) Sequences and Series

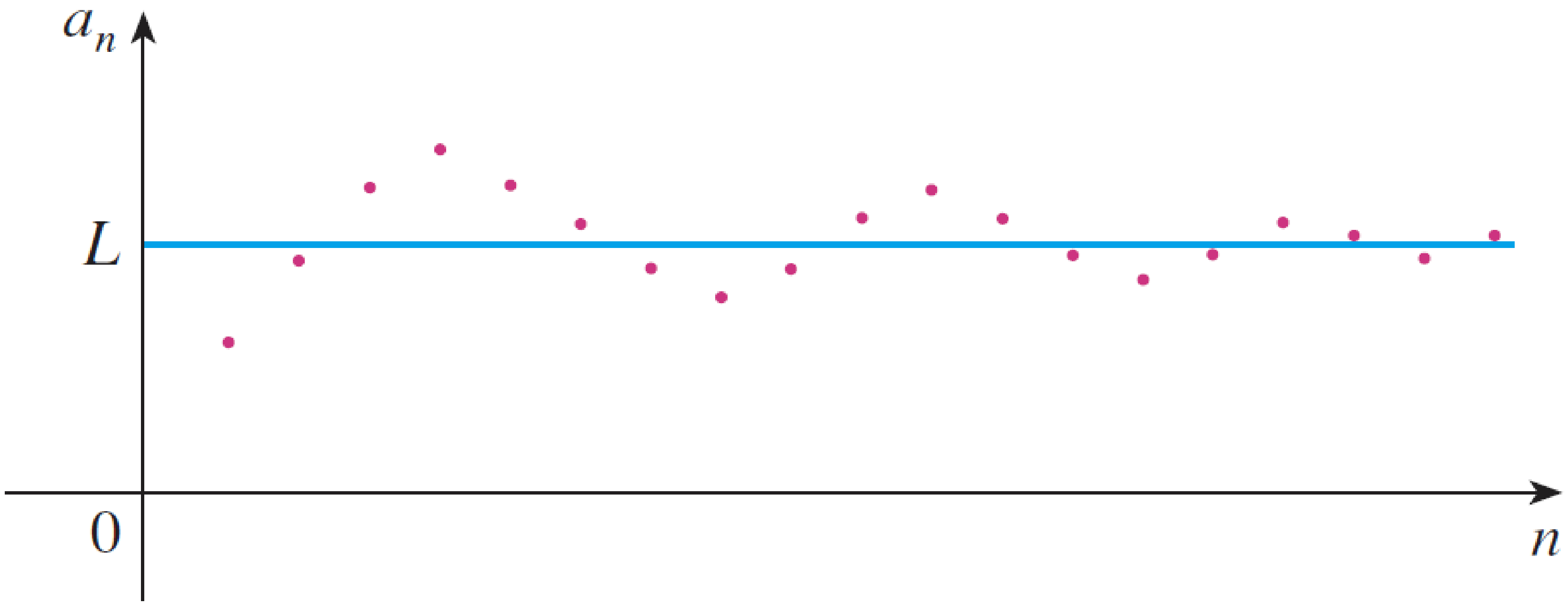
(実)数列と級数

Notice: In all lecture notes, the contents marked with * are not in the scope of the final examination.

*9.1 Review of (Real) Sequences and Series

Let's review Calculus II (微積分 II) (for real number).

Sequences (実数)数列



*9.1 Review of (Real) Sequences and Series

A **sequence:** a list of numbers written in a definite order

$$\{a_1, a_2, a_3, a_4, \dots, a_n, \dots\}$$

The first term

The second term

The nth term

Notice that each term a_n will have a successor a_{n+1}.

Notice that **for every positive integer there is a corresponding number** and **so a sequence can be defined as a function whose domain is the set of positive integers**.
But **we usually write** a_n instead of the function notation $f(n)$ for the value of the function **at the number** n.

Notation The **sequence** $\{a_1, a_2, a_3, a_4, \dots, a_n, \dots\}$ is also denoted by

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

*9.1 Review of (Real) Sequences and Series

Example Some sequences can be defined by giving a formula for the nth term.

	1. Preceding notation	2. Defining notation	3. Writing out the terms of the sequences
(a)	$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$	$a_n = \frac{n}{n+1}$	$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$
(b)	$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$	$a_n = \frac{(-1)^n(n+1)}{3^n}$	$\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$
(c)	$\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3},\ n \geqslant 3$	$\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$
(d)	$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$	$a_n = \cos\frac{n\pi}{6},\ n \geqslant 0$	$\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\right\}$

*9.1 Review of (Real) Sequences and Series

In general, the **notation**

$$\lim_{n\to\infty} a_n = L$$

means that **the terms of the sequence** $\{a_n\}$ **approach** L **as** n **becomes large**.

Definition: A **sequence** $\{a_n\}$ **has the limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**).

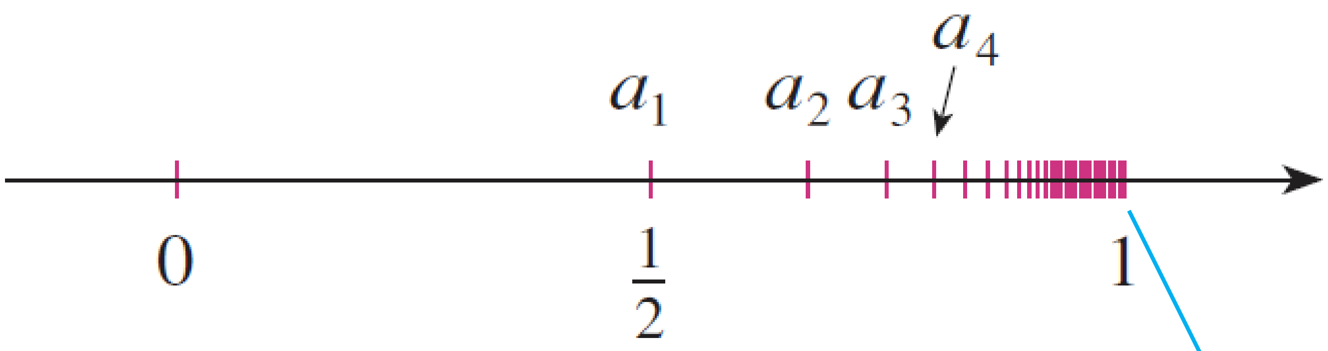
Otherwise, we say the sequence **diverges** (or is **divergent**).

*9.1 Review of (Real) Sequences and Series

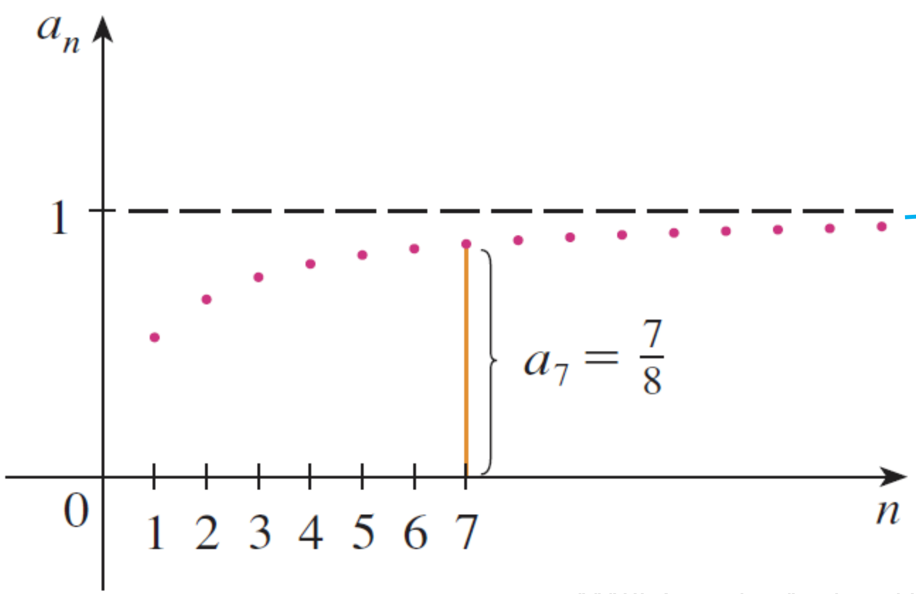
Convergent sequence example:

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$$

On a number line



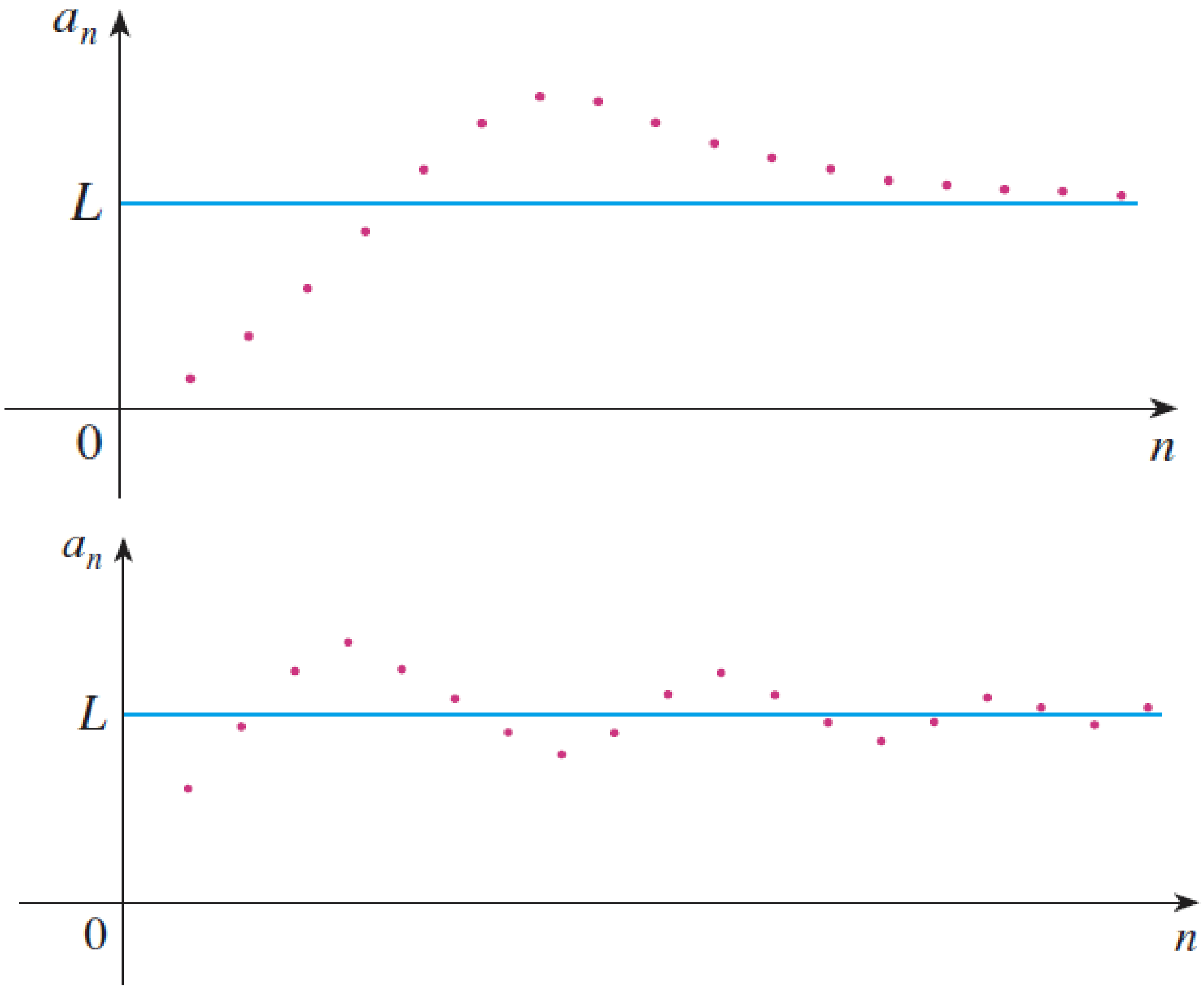
Plotting a graph



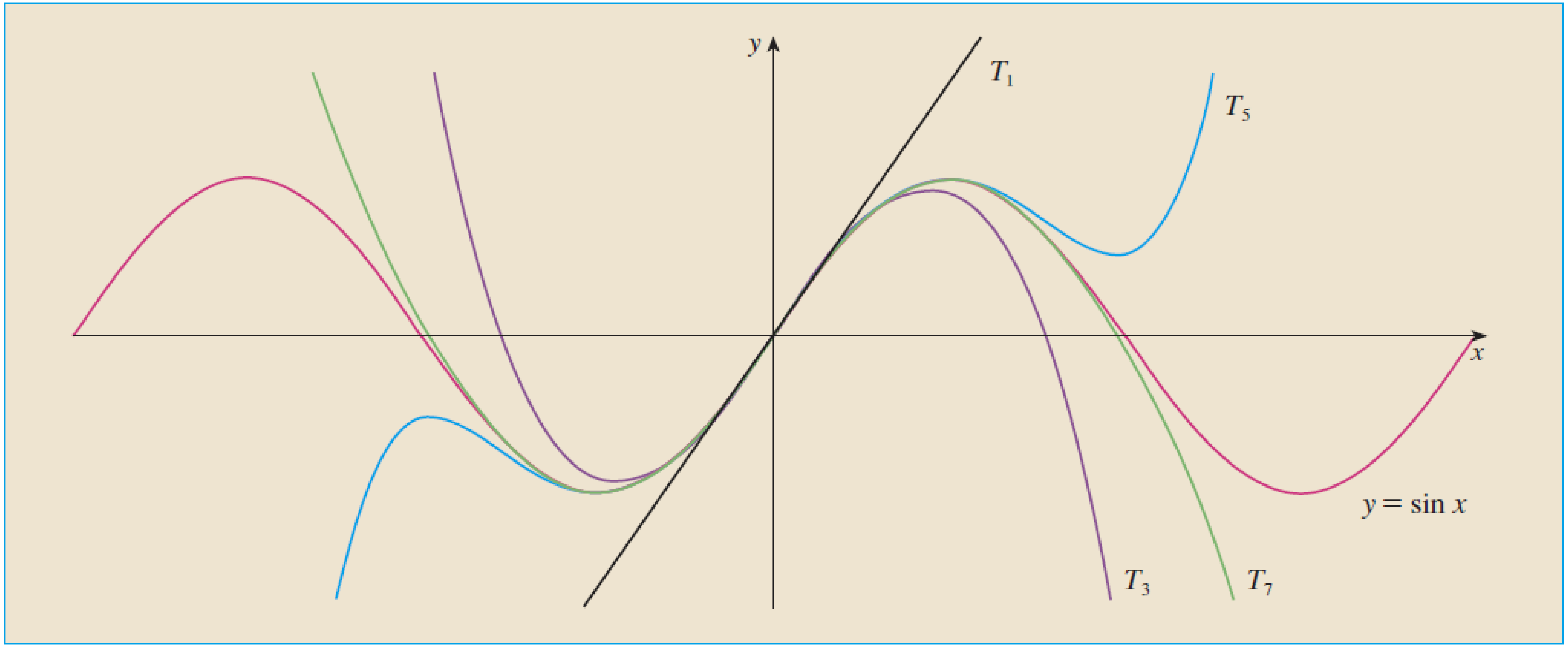
$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

*9.1 Review of (Real) Sequences and Series

Convergent sequence example:



Series (実数)級数



The partial sums T_n of a Taylor series provide better and better approximations to a function a n increases.

*9.1 Review of (Real) Sequences and Series

Calculate the addition for the terms of an **infinite sequence** $\{a_n\}_{n=1}^{\infty}$, we get

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

*9.1 Review of (Real) Sequences and Series

From $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$

a_n

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots$

$\sum a_n$

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

We can observe that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

*9.1 Review of (Real) Sequences and Series

Consider the partial sums

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and in general

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

*9.1 Review of (Real) Sequences and Series

Definition: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denotes its nth **partial sum**:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence $\{s_n\}$ is **convergent** and $\lim_{n\to\infty} s_n = s$ exists as a **real number**, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. We can see $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$.

Otherwise, the series is called **divergent**.

*9.1 Review of (Real) Sequences and Series

Definition: The (real) geometric series (幾何級数)

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is **convergent** if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

if $|r| \geq 1$, then **geometric series** is **divergent**.

$\mathbf{9.2}$ (Complex) Sequences and Series

(複数)数列と級数

Sequences (数列)

A **sequence** $\{z_n\}$, where $n = 1, 2, 3, \ldots,$ is **a function** whose **domain is the set of positive integers** and **whose range is a subset of the complex numbers C**.

For example, the sequence $\{1 + i^n\}$ is

$$1 + i, \qquad 0, \qquad 1 - i, \qquad 2, \qquad 1 + i, \qquad \ldots \tag{6.1.1}$$

9.2 (Complex) Sequences and Series (数列と級数)

Sequences (数列)

If $\lim_{n\to\infty} z_n = L$, where L is a complex number, we say the sequence $\{z_n\}$ is **convergent (収束する).**

Sequence that is **not convergent** is said to be **divergent (発散する).**

$\{z_n\}$ converges to the number L, if for each positive real number ε, an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.

Since $|z_n - L|$ is distance, the terms z_n of a sequence that converges to L can be made arbitrarily close to L.

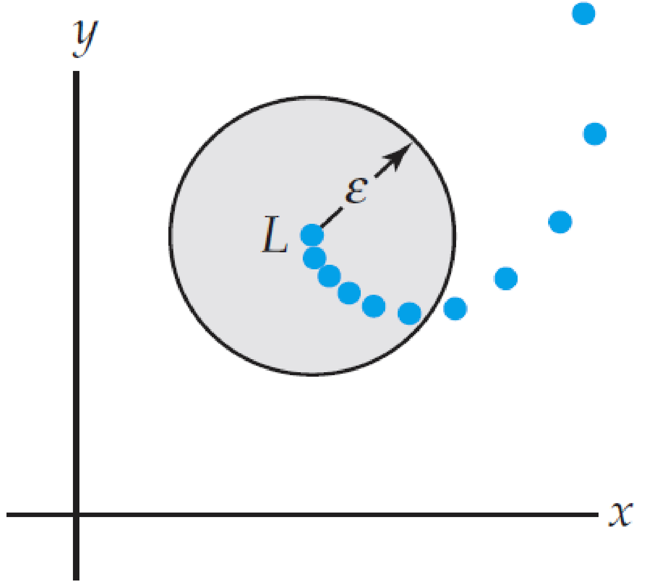


Figure 6.1 If $\{z_n\}$ converges to L, all but a finite number of terms are in every ε-neighborhood of L.

Sequences (数列)

For example, the sequence $\{1+i^n\}$

$$
\begin{array}{cccccc}
1+i, & 0, & 1-i, & 2, & 1+i, & \dots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
n=1, & n=2, & n=3, & n=4, & n=5, &
\end{array}
$$

The sequence $\{1+i^n\}$ **is divergent** because **the general term** $z_n = 1 + i^n$ **does not approach a fixed complex number** as $n \to \infty$.

EXAMPLE (例題) 6.1.1 A Convergent Sequence

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ **converges or not.**

Solution (解答):

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ **converges** since $\lim_{n\to\infty} \frac{i^{n+1}}{n} = 0$. As we see from

$$-1, -\frac{i}{2}, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5}, \cdots,$$

and Figure 6.2, the terms of the sequence, marked by colored dots in the figure, **spiral in toward the point** $z = 0$ as n increases.

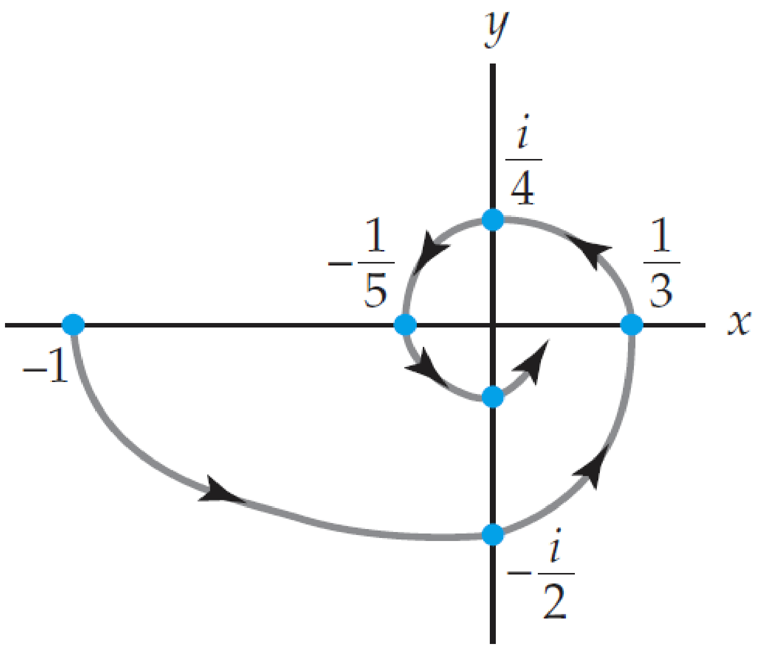


Figure 6.2 The terms of the sequence $\{\frac{i^{n+1}}{n}\}$ spiral in toward 0.

Sequences (数列)

Theorem 6.1 Criterion (基準) for Sequence Convergence

Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $L = x + iy$. Then

$$\lim_{n\to\infty} z_n = L$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

This theorem for sequences is the analogue of Theorem 2.1 in Lecture 2.

Theorem 2.1 Real and Imaginary Parts (実部と虚部) of a Limit

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, and

$L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ **if and only if**

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Additional EXAMPLE (例題) 1 Using Theorem 6.1

The sequence $\left\{\frac{1}{n^3} + i\right\}$ converges or not.

Solution (解答):

The sequence $z_n = \frac{1}{n^3} + i \quad (n = 1,2,\ldots)$ converges to i since

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i\lim_{n\to\infty} 1 = 0 + i\cdot 1 = i$$

For each number $\varepsilon > 0$

$$|z_n - i| = \frac{1}{n^3} < \varepsilon \quad \text{whenever } n > \frac{1}{\sqrt[3]{\varepsilon}}$$

Sequences (数列)

EXAMPLE (例題) 6.1.2 Using Theorem 6.1

The sequence $\left\{\frac{3+ni}{n+2ni}\right\}$ converges or not.

Solution (解答):

From $z_n = \dfrac{3+ni}{n+2ni} = \dfrac{(3+ni)(n-2ni)}{(n+2ni)(n-2ni)} = \dfrac{(3+ni)(n-2ni)}{n^2+4n^2} = \dfrac{2n^2+3n}{5n^2} + i\,\dfrac{n^2-6n}{5n^2}$

we see that when $n \to \infty$

$$\lim_{n\to\infty} \mathrm{Re}(z_n) = \lim_{n\to\infty} \frac{2n^2+3n}{5n^2} = \lim_{n\to\infty}\left(\frac{2}{5}+\frac{3}{5n}\right) = \frac{2}{5}$$

$$\lim_{n\to\infty} \mathrm{Im}(z_n) = \lim_{n\to\infty} \frac{n^2-6n}{5n^2} = \lim_{n\to\infty}\left(\frac{1}{5}-\frac{6}{5n}\right) = \frac{1}{5}$$

From Theorem 6.1, the results are sufficient to conclude that the given sequence converges to $L = x + iy = \frac{2}{5} + \frac{1}{5}i$.

Series (級数)

An infinite series or series of complex numbers

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots$$

is **convergent** if the **sequence of partial sums (部分和)** $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges.

If $S_n \to L$ as $n \to \infty$, we say that **the series converges to** L or that **the sum of the series** is L.

Additional EXAMPLE (例題) 2

Show that if $\sum_{n=1}^{\infty} z_n = L$ where $L = x + iy$, then $\sum_{n=1}^{\infty} \bar{z}_n = \bar{L}$.

Solution (解答):

We write $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$.

First of all, we note that

$$\sum_{n=1}^{\infty} x_n = x \text{ and } \sum_{n=1}^{\infty} y_n = y$$

Then since $\sum_{n=1}^{\infty}(-y_n) = -y$, it follows that

$$\sum_{n=1}^{\infty} \bar{z}_n = \sum_{n=1}^{\infty}(x_n - iy_n) = \sum_{n=1}^{\infty}[x_n + i(-y_n)] = x - iy = \bar{L}$$

Geometric Series (幾何級数)

A **geometric series** is any series of the form

$$\sum_{n=1}^{\infty} az^{n-1} = a + az + az^2 + \cdots \tag{6.1.2}$$

For (6.1.2), the nth term of **the sequence of partial sums** is

$$S_n = a + az + az^2 + \cdots + az^{n-1} \tag{6.1.3}$$

Geometric Series (幾何級数)

When **an infinite series is a geometric series**, it is always possible to find a formula for S_n.

Why? We can multiply S_n in (6.1.3) by z,

$$zS_n = az + az^2 + az^3 + \cdots + az^n$$

and subtract this result from S_n, then we have

$$S_n - zS_n = (a + az + az^2 + \cdots + az^{n-1}) - (az + az^2 + az^3 + \cdots + az^n)$$

$$(1 - z)S_n = a - az^n$$

$$\Rightarrow \qquad S_n = \frac{a(1 - z^n)}{1 - z} \tag{6.1.4}$$

Now $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$, and so $S_n \to \frac{a}{1-z}$.

In other words, for $|z| < 1$ **the sum of a geometric series** (6.1.2) is $\frac{a}{1-z}$ (i.e. **convergent**):

$$\sum_{n=1}^{\infty} az^{n-1} = a + az + az^2 + \cdots = \frac{a}{1-z} \tag{6.1.5}$$

A geometric series (6.1.2) **diverges** when $|z| \geq 1$.

Special Geometric Series

If we set $a = 1$, the equality in (6.1.5) is

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots \tag{6.1.6}$$

If we then replace the symbol z by $-z$ in (6.1.6), we get a similar result

$$\frac{1}{1 + z} = 1 - z + z^2 - z^3 + \cdots \tag{6.1.7}$$

Like (6.1.5), the equality in (6.1.7) is valid for $|z| < 1$ since $|-z| = |z|$. Now with $a = 1$, (6.1.4) gives us the sum of the first n terms of the series in (6.1.6):

$$\frac{1 - z^n}{1 - z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$$

EXAMPLE (例題) 6.1.3 Convergent Geometric Series

The series $\sum_{n=1}^{\infty} \frac{(1+2i)^n}{5^n}$ is convergent or divergent?

Solution (解答):

The infinite series

$$\sum_{n=1}^{\infty} \frac{(1+2i)^n}{5^n} = \underbrace{\frac{1+2i}{5}}_{a} + \underbrace{\frac{(1+2i)^2}{5^2}}_{az} + \underbrace{\frac{(1+2i)^3}{5^3}}_{az^2} + \cdots$$

is a geometric series. It has the form given in (6.1.2) with $a = \frac{1+2i}{5}$ and $z = \frac{1+2i}{5}$. Since $|z| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{\sqrt{5}}{5} < 1$, the series is **convergent** and its sum is given by (6.1.5):

$$\sum_{n=1}^{\infty} \frac{(1+2i)^n}{5^n} = \frac{\overbrace{\frac{1+2i}{5}}^{a}}{1 - \underbrace{\frac{1+2i}{5}}_{z}} = \frac{1+2i}{4-2i} = \frac{1}{2}i$$

Theorem 6.2 A Necessary Condition for Convergence

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Proof

Let L denote the sum of the series. Then $S_n \to L$ and $S_{n-1} \to L$ as $n \to \infty$.

By taking the limit of both sides of $S_n - S_{n-1} = z_n$ as $n \to \infty$, we have

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} z_n$$

$$L - L = \lim_{n\to\infty} z_n$$

$$0 = \lim_{n\to\infty} z_n$$

we obtain the desired conclusion. ■

9.3 Series Tests

A Test for Divergence

Theorem 6.3 The nth Term Test for Divergence

If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

For example,

the series $\sum_{n=1}^{\infty} \frac{in+5}{n}$ **diverges** since $z_n = \frac{in+5}{n} \to i \neq 0$ as $n \to \infty$.

The **geometric series** (6.1.2) **diverges** if $|z| \geq 1$ because even in the case when $\lim_{n\to\infty} |z_n|$ exists, **the limit is not zero**.

Definition 6.1 Absolute and Conditional Convergence (絶対収束と条件収束)

An infinite series $\sum_{n=1}^{\infty} z_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty}|z_n|$ converges. An infinite series $\sum_{n=1}^{\infty} z_n$ is said to be **conditionally convergent** if it converges but $\sum_{n=1}^{\infty}|z_n|$ diverges.

p-series

In elementary calculus **a <u>real-value</u> series** of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series and **converges for** $p > 1$ and **diverges for** $p \leq 1$.

EXAMPLE (例題) 6.1.4 Absolute Convergence

The series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ is absolute convergent or not.

Solution (解答):

The series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ is absolutely convergent since the series $\sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right|$

is the same as the real convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Here we identify $p = 2 > 1$.

As in Real-value calculus:

Absolute convergence implies convergence.

We can therefore conclude that the series in Example 6.1.4,

$$\sum_{n=1}^{\infty} \frac{i^n}{n^2} = i - \frac{1}{2^2} - \frac{i}{3^2} + \frac{1}{4^2} + \cdots$$

converges because it is **absolutely convergent.**

Tests for Convergence

Theorem 6.4 Ratio Test

Suppose $\sum_{n=1}^{\infty} z_n$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L \tag{6.1.9}$$

(i) If $L < 1$, then **the series converges absolutely**.

(ii) If $L > 1$ or $L = \infty$, then **the series diverges**.

(iii) If $L = 1$, the test is inconclusive (i.e. no idea about the result).

Tests for Convergence

Theorem 6.5 Root Test

Suppose $\sum_{n=1}^{\infty} z_n$ is a series of complex terms such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L \qquad (6.1.10)$$

(i) If $L < 1$, then **the series converges absolutely.**

(ii) If $L > 1$ or $L = \infty$, then **the series diverges.**

(iii) If $L = 1$, **the test is inconclusive.**

Review for Lecture 9

- (Complex) Sequences and Series
- Convergence and Divergence
- Geometric Series
- p-series
- Absolute and Conditional Convergence
- Series Tests

Exercise

Please Check http://web-ext.u-aizu.ac.jp/~xiangli/teaching/MA06/index.html

References

[1] A First Course in Complex Analysis with Application, Dennis G. Zill and Patrick D. Shanahan, Jones and Bartlett Publishers, Inc. 2003

[2] Wikipedia