A Dual Formulation of The Traffic Assignment Problem for OD-matrix Estimation

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ABSTRACT

Congestion, accidents, greenhouse gas emission and others seem to become unsolvable problems for all levels of management in modern large cities worldwide. The increasing dynamics of motorization requires development of innovative methodological tools and technical devices to cope with problems emerging in the road networks. Primarily, control system for urban traffic area has to be created to support decision makers by processing a big volume of transportation data. The input for such a system is a volume of travel demand between origins and destinations — OD-matrix. The present work is devoted to the problem of OD-matrix estimation. The original technique of OD-matrix estimation is offered by virtue of plate scanning sensors location. Mathematically developed technique is based on a dual formulation of the traffic assignment problem (equal journey time by alternative routes between any OD-pair). Traffic demand between certain OD-pair is estimated due to journey time obtained from plate scanning sensors. Moreover, the functional relationship between traffic demand and journey time is obtained explicitly for the network of parallel routes with one OD-pair. Eventually, the developed method has been tested on the experimental data of the Saint-Petersburg road network.

Categories and Subject Descriptors

G.1.6 [**Mathematics of Computing**]: Numerical analysis—*optimization*

General Terms

Theory

Keywords

OD-matrix estimation, traffic assignment problem, duality theory

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1. INTRODUCTION

OD-matrix estimation and reconstruction are urgent and complicated challenges, since road networks of modern cities are extremely large and intricate. In general, OD-matrix estimation and reconstruction are different problems: the first means to obtain approximate values, while the second means to obtain precise values of an actual traffic demand [1]. One of the first mathematical models for OD-matrix estimation was formulated in a form of bi-level program [2]. Despite numerous publications, this problem still attracts researchers from all over the world [3–8]. A detailed comparative analysis of the methods for trip matrix estimation was made in [4]. From a practical perspective, the most promising technique for trip matrix estimation is combination of data obtained both from plate scanning sensors and link-flow counts [5].

This paper is also devoted to the problem of OD-matrix estimation. We believe that a plate scanning sensor is the highly efficient engineering equipment. Indeed, due to link-flow counts one could obtain solely amount of vehicles on the link, while plate scanning allows estimating the average travel time between origin and destination by identification the vehicle in the origin and destination points. Since the travel time between an origin-destination pair is a Lagrange multiplier for a primal traffic assignment problem (TAP), it is the variable in a dual formulation of TAP. Therefore, we are able to formulate a new bi-level optimization program for OD-matrix estimation based on data from link-flow plate scanning sensors on congested networks.

The rest of this paper is organized as follows. In Section 2 the network of parallel routes with one OD-pair is investigated. The idea of OD-matrix estimation based on information about travel times between OD-pairs is clarified. Section 3 provides a dual formulation of the traffic assignment problem for a general topology network. Section 4 describes a bi-level optimization program for OD-matrix estimation on a congested network by virtue of plate scanning sensors. Section 5 is devoted to the experimental implementation of the developed approach to the Saint-Petersburg road network. Conclusions are given in Section 6.

2. THE NETWORK OF PARALLEL ROUTES

Let us consider a transportation network presented by a digraph with one OD-pair. Let us introduce the following notation: F is the traffic demand between OD-pair; f_i is the traffic flow on the route i, $i = \overline{1,n}$, $f = (f_1, \ldots, f_n)$,

$\sum_{i=1}^{n} f_i = F$; $t_i(f_i) = a_i + b_i f_i$ is the travel time on congested arc i, $i = \overline{1,n}$. In the present work we model the travel time on the congested arc as the linear function.

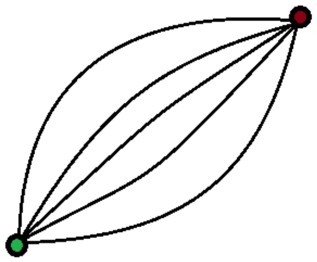

Figure 1: The road network of parallel routes with one OD-pair

Let us formulate the traffic assignment problem on the network of parallel routes as an optimization program [9, 10]:

$$z(f^*) = \min_f z(f) = \min_f \sum_{i=1}^{n} \int_0^{f_i} t_i(u)du, \qquad (1)$$

with constraints

$$\sum_{i=1}^{n} f_i = F, \qquad (2)$$

$$f_i \geqslant 0 \quad \forall i = \overline{1,n}. \qquad (3)$$

Wardrop's first principle states that the journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route [10, 11]. The traffic flows that satisfy this principle are usually referred to as "user equilibrium" (UE) flows, since each user chooses the route that is the best. On the network of parallel routes UE is reached by such assignment $f^* = (f_1^*, \ldots, f_n^*)$ as:

$$\begin{cases} t_i(f_i^*) = t^* > 0 & \text{when } f_i^* > 0, \\ t_i(f_i^*) > t^* & \text{when } f_i^* = 0, \end{cases} \quad i = \overline{1,n}.$$

Thus, the mathematically formalized idea of UE (1)–(3) can be used in reconstruction of traffic assignment on the network between the origin-destination pair. However, if it is the travel time t^* between OD-pair that is known, we are able to reconstruct traffic demand F on the linear network of parallel routes.

Without loss of generality we assume that the routes are numbered as follows:

$$a_1 \leqslant \ldots \leqslant a_n.$$

THEOREM 1. *The traffic demand F for a linear network of parallel routes can be obtained explicitly:*

$$F = t^* \sum_{s=1}^{k} \frac{1}{b_s} - \sum_{s=1}^{k} \frac{a_s}{b_s}, \qquad (4)$$

where k satisfies

$$a_1 \leqslant \ldots a_k < t^* \leqslant a_{k+1} \ldots \leqslant a_n. \qquad (5)$$

PROOF. The travel time t^* through used routes is the Langrangian multiplier that corresponds to the restriction (2) of optimization program (1)–(3) [9, 12, 13]. According to [9] the following relation holds:

$$t^* = \frac{F + \sum_{s=1}^{k} \frac{a_s}{b_s}}{\sum_{s=1}^{k} \frac{1}{b_s}},$$

and, hence, (4) follows directly when k satisfies (5). $\square$

Therefore, if we know travel time of the vehicle on any of alternative routes between the OD-pair, the appropriate traffic demand can be uniquely reconstructed. Due to such results the developed approach seems to be promising. The main idea of the method based on the first principle of Wardrop: if we define the journey time of the vehicle on any of actually used routes between certain OD-pair, then we believe that the journey time on all other used routes is the same.

3. DUAL FORMULATION OF TAP

Let us consider the network of general topology presented by graph $G = (N, A)$. We introduce the following notation: W is the set of OD-pairs, $w \in W$, $W \in N$; K^w is the set of routes connecting OD-pair w; F^w is the traffic demand for OD-pair w, $F = \left(F^1, \ldots, F^{|W|}\right)^{\mathrm{T}}$; f_k^w is the traffic flow on the route $k \in K^w$, $\sum_{k \in K^w} f_k^w = F^w$; $f^w = \{f_k^w\}_{k \in K^w}$ and $f = \{f^w\}_{w \in W}$; x_a is the traffic flow on the arc $a \in A$, $x = (\ldots, x_a, \ldots)$; $t_a(x_a)$ is the link travel cost on the arc $a \in A$; $\delta_{a,k}^w$ is the indicator: 1 if the acr a is included in the route k, 0 otherwise.

User equilibrium on the transportation network G is reached by such x^* that

$$Z(x^*) = \min_x \sum_{a \in A} \int_0^{x_a} t_a(u)du, \qquad (6)$$

subject to

$$\sum_{k \in K^w} f_k^w = F^w, \quad \forall w \in W, \qquad (7)$$

$$f_k^w \geqslant 0, \quad \forall w \in W, \qquad (8)$$

$$(9)$$

with definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A. \qquad (10)$$

User equilibrium principle allows us to introduce t_w^*, that is equilibrium journey time for any OD-pair w.

Proposition. t_w^* is the Lagrange multiplier in the optimization program (6)–(10) corresponding to the constraint (8).

Proof. The Lagrangian of the problem (6)–(10) is

$$L = \sum_{a \in A} \int_0^{x_a} t_a(u)du + \sum_w \mu_w \left(F^w - \sum_{k \in K^w} f_k^w\right) + \\ + \sum_w \sum_{k \in K^w} \eta_k^w \left(-f_k^w\right),$$

where μ_w and $\eta_k^w \geq 0$ are Lagrangian multipliers, and differentiation of the Lagrange]ian yields:

$$\frac{\partial L}{\partial f_k^w} = \sum_{a \in k} t_a(x_a) - \mu_w - \eta_k^w = 0.$$

Note, that according to complementary slackness $\eta_k^w f_k^w = 0$. Thus, for $f_k^w > 0$ the following expression holds

$$\sum_{a \in k} t_a(x_a) = \mu_w, \quad \forall k \in K^w, w \in W \tag{11}$$

Actually, the left part of (11) is the journey time on any used route ($f_k^w > 0$) between OD-pair r. Therefore, proposition is proved.

Eventually, according to the proposition the following equality is true:

$$t_w^* = \sum_{a \in k} t_a(x_a) \quad \forall k \in K^w, w \in W.$$

We introduced multipliers $T = (t_1, \dots, t_{|W|})^{\mathrm{T}}$ for the constraints (7), and define the dual traffic equilibrium problem:

$$\max \theta(T)$$

where $\theta(T)$ is defined by

$$\theta(T) = \min_{f \geqslant 0} \left\{ \sum_{a \in A} \int_0^{x_a} t_a(s)ds + \sum_r t_r \left(F^r - \sum_{k \in K^r} f_k^r \right) \right\},$$

subject to definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A.$$

4. OD-MATRIX ESTIMATION FROM PLATE SCANNING SENSORS

Link-flow counts provide the amount of vehicles on the links. Plate scanning sensors associated with the certain links identify plates of vehicles from link-flow. Thus, when any vehicle crosses a link with some sensor then sensor records its plate number and fixation time. Eventually, database consisting of {plate number, fixation time, number of sensor} is accumulated [3]. With the help of such database, the travel time between any origin-destination pair can be directly evaluated. Indeed, one just has to know fixation time of the vehicle in origin and fixation time in destination to define t_r^* for any r.

Therefore, the following bi-level optimization program can be formulated:

$$\min_F \left(\overline{F} - F\right)^{\mathrm{T}} U^{-1} \left(\overline{F} - F\right) + (T^* - T)^{\mathrm{T}} (T^* - T), \tag{12}$$

subject to

$$F \geq 0, \tag{13}$$

where T solves

$$\max \theta(T), \tag{14}$$

where $\theta(T)$ is defined by

$$\theta(T) = \min_{f \geqslant 0} \left\{ \sum_{a \in A} \int_0^{x_a} t_a(s)ds + \sum_r t_r \left(F^r - \sum_{k \in K^r} f_k^r \right) \right\}, \tag{15}$$

subject to definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A. \tag{16}$$

Here, (12) is the generalized least squares estimation and $\overline{F}$ is the aprior volume of travel demand between all OD-pairs, and U is the weighting matrix.

5. COMPUTATIONAL EXPERIMENT

Let us consider the road network of Saint-Petersburg (fig. 2). We define seven origin-destination pairs with seven shortest routes from seven periphery origins {1,2,3,4,5,6,7} to the center destination {8}. According to STSI (State Traffic Safety Inspectorate), nowadays there are 253 plate scanning sensors observing the road network of Saint-Petersburg (fig. 3). Due to these sensors, we are able to identify travel time between chosen OD-pairs (table 1). The developed approach is based on the user equilibrium principle, which suggests that the value of travel time on the shortest route is

Figure 2: Selected OD-pairs on the Saint-Petersburg road network with the shortest routes

Figure 3: Sensors location on the Saint-Petersburg road network

Table 1: Journey times obtained from plate scanning sensors

Route between OD-pair	Travel time t^* (minutes)
1–8	89
2–8	80
3–8	83
4–8	78
5–8	45
6–8	57
7–8	36

the travel time on any actually used route. Moreover, we are able to calculate an aprior flow $\overline{F}$ using the gravity model [4].

Let us use these data as inputs for bi-level optimization program (12)–(16). MATLAB was employed to carry out the simulation. The results of simulation are provided in the table 2. Moreover, these results are available in comparison with aprior flows. Fig. 4 gives a visualization of such a comparison. One can see that rough aprior estimation of trip flows, obtained by gravity model, was adjusted by virtue of information about actual travel time between OD-pairs. Therefore, developed in this paper approach seems to be quite useful.

Table 2: Comparison of model flow with aprior flow

OD-pair	Aprior flow	Model flow
1–8	5523	5910
2–8	12232	11253
3–8	6827	6295
4–8	6938	7631
5–8	5534	5080
6–8	4254	4650
7–8	3395	3202

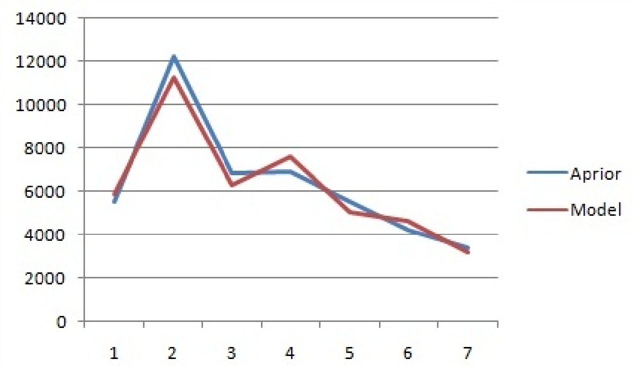

Figure 4: Comparison of model flow with aprior flow

6. CONCLUSION

The paper was devoted to the problem of OD-matrix estimation. The original technique of OD-matrix estimation based on a dual formulation of the traffic assignment problem was offered. Traffic demand between certain OD-pair was estimated due to the journey time obtained from plate scanning sensors. Moreover, the functional relationship between the traffic demand and the journey time was obtained explicitly for the network of parallel routes with one OD-pair. Eventually, the developed method has been tested on the experimental data of the Saint-Petersburg road network.

7. REFERENCES

[1] Hazelton M. Inference for origin-destination matrices: estimation, prediction and reconstruction // Transportation Research Part B. 2001. No 35. P. 667–676.

[2] Yang H., Sasaki T., Iida Y., Asakura Y. Estimation of origin-destination matrices from link traffic counts on congested networks // Transportation Research Part B. 1992. No 26 (6). P. 417–434.

[3] Zakharov V., Krylatov A. OD-matrix estimation based on plate scanning // 2014 International Conference on Computer Technologies in Physical and Engineering Applications (ICCTPEA). 2014. P. 209-210.

[4] Medina A., Taft N., Salamatian K., Bhattacharyya S., Diot C. Traffic matrix estimation: existing techniques and new directions // Computer Communication Review – Proceedings of the 2002 SIGCOMM conference. 2002. No 32. P. 161–174.

[5] Castillo E., Menedez J. M., Jimenez P. Trip matrix and path flow reconstruction and estimation based on plate scanning and link observations // Transportation Research Part B. 2008. No 42. P. 455–481.

[6] Bianco L., Cerrone C., Cerulli R., Gentili M. Locating sensors to observe network arc flows: exact and heuristic approaches // Computers and Operation Research. 2014. No 46. P. 12–22.

[7] Minguez R., Sanchez-Cambronero S., Castillo E., Jimenez P. Optimal traffic plate scanning location for OD trip matrix and route estimation in road networks // Transportation Research Part B. 2010. No 44. P. 282–298.

[8] Bierlaire M. The total demand scale: a new measure of quality for static and dynamic originŰdestination trip tables // Transportation Research Part B. 2002. No 36. P. 282–298.

[9] Krylatov A. Yu. Optimal strategies for traffic flow management on the transportation network of parallel links // Vestnik of St. Petersburg State University. Series 10. Applied Mathematics. Computer Science. Control Processes. 2014. No 2. P. 121–130.

[10] Patriksson M. The Traffic Assignment Problem: Models and Methods. Utrecht, Netherlands: VSP, 1994. 223 p.

[11] Wardrop J. G. Some theoretical aspects of road traffic research // Proc. Inst. Civil Eng. 1952. Vol. 1, No 3. P. 325–362.

[12] Zakharov V. V., Krylatov A. Yu. Transit Network Design for Green Vehicles Routing // Advances in Intelligent Systems and Computing. 2015. Vol. 360. P. 449–458.

[13] Zakharov V. V., Krylatov A. Yu. Competitive routing of traffic flows by navigation providers // Automation and Remote Control, 2016. Vol. 77, No 1. P. 179–189.