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Manual of ope.math: a program for the calculation of operator product expansions in free field realizations of conformal field theory (Version 1.0/April 30, 1993)

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ABSTRACT

We develop a package "ope.math" for *Mathematica* to compute operator product expansions (OPE) of composite operators (currents) in terms of free fields in conformal field theories. Further, we can also calculate OPEs of currents expressed by vertex operators. We give examples of free field realizations of current algebras, Hamiltonian reduction of current algebras and BRST cohomology of two dimensional gravity coupled to $c \leq 1$ minimal models.

1. Introduction

Operator product expansion (OPE) is one of the standard technique in conformal field theories (CFT) [1] to compute loop algebras such as affine (super-) Lie algebras, W-algebras, parafermionic current algebras, superconformal algebras, etc in a compact form. Every such algebra can (or is expected to) be realized by using free fields, and the representation theory of such an algebra can be resolved as the Felder's BRST cohomology [2].

Because the complication in computing OPEs of composite operators increases quickly, we need some tools to obtain easily the result of OPEs. We present a program "ope.math" that makes it possible to compute the OPE of currents expressed in terms of free fields and vertex operators. In the proposed program "ope.math", we only treat free fermions, free bosons, and vertex operators. If one wants to study a non-Abelian algebra, e.g. $sl(2)$, one should construct the algebra by using the method of Wakimoto realization [3], in which the currents are realized in terms of differential polynomials of free fields.

The highest weight vectors are constructed by using vertex operators. We can discuss the representation theory of the algebra by the vertex operator construction of highest weight vectors. In the version 1.0, we cannot compute N($>$ 2)-point correlation functions of the vertex operators. We are, however, planning to do that.

We have already had Thielemans's package "OPEdefs.m" [4]. His package treats not only free fields but also non-Abelian currents themselves based on the associativity [5]. In our package, we do not treat non-Abelian currents, and we construct them by the free field realization, though it becomes, however, quickly complicated. Thus, since the associativity is trivially satisfied for free fields, the program of the package is very simple.

Though his package can potentially treat vertex operators, the consideration about the cocycle factor is not found in [4]. We consistently introduce the vertex operator and the cocycle factor in our package. We discuss the property of the cocycle factor which is taken into account in the package. Because we do not, however, know how to evaluate the cocycle factor completely in general, the user will have to evaluate them in particular cases.

The method to compute OPEs in "ope.math" is basically the Wick expansion. Among free fields, the Wick expansion simply means separating the singular part and the regular part of the operator product. Applying this to the operator product of currents including vertex operators, we have a rule to compute the OPE, and we realize it in the package by first computing the vertex part of operators.

The computation of algebras by using *Mathematica* [6] is one of the easiest ways. The pattern matching of functions is most likely the way we calculate algebras in a notebook by hand. After starting *Mathematica*, we only need to execute

```
In[1] :=   << ope.math
```

Then, we can use "operators", by which we mean the elements of operators, `psi[]`, `bi[]`, `ci[]`, `be[]`, `ga[]`, `pj[]`, `vp[]`, `ro[]`, `xi[]`, `Nl[]` and their polynomials, and "functions", by which we mean mapping operators to operators, `ope[]`, `res[]`, `der[]`, `Fsort[]`, `Texp[]` and `Current[]`, which are defined in "ope.math" as will be explained later.

Special composite operators are also prepared from the beginning. They are stress tensors and $u(1)$ currents, `Tps[]` for `ps[]`, `Tbc[]` and `Jc[]` for `bi[]` and `ci[]`, `Tbg[]` and `Jg[]` for `be[]` and `ga[]`, `Tpj[]` and `Jp[]` for `pj[]`.

For an advanced use like constructing spin operators, a function `bosonize[]` is prepared. To use the function `bosonize[]`, one must set bosonization rules for every elementary field and every vertex operator by using function `BRule[]`. Then, the function `bosonize[]` translates all composite operators to the bosonized forms.

This manuscript is organized as follows. In sec.2., we give the definition of free fields and the normal ordered product of them. In sec.3., we explain the function `ope` and the other functions. In sec.4., we will explain the vertex operator and the cocycle factor. In sec.5., the function of bosonization is defined. Simple examples are given in the subsections in order to understand how to use the functions in the package. Sec.6. is devoted to show the examples of the calculations in the recent development in the study of two-dimensional gravity. In sec.6.1., we construct a free field realization of the affine Lie superalgebra $osp(2|2)$. In sec.6.2., we show the Hamiltonian reduction to give topological gravity from the Lie superalgebra $osp(2|2)$. In sec.6.3., following to Kanno and Sarmadi [7], we discuss two-dimensional gravity coupled to minimal models.

2. Free fields and composite fields

We explain how to make a composite operator in terms of free fields. Composite operators considered here are differential polynomials of free fields.

2.1. Free Fields

First of all, we introduce free fields which we can use in the program "ope.math".

j-th free field	$f[derivative, j, position]$
Majorana fermion $\partial^i\psi_j(z)$	`ps[i,j,z]`
complex fermions $(\partial^i b_j(z), \partial^i c_j(z))$	`bi[i,j,z], ci[i,j,z]`
complex bosons $(\partial^i \beta_j(z), \partial^i \gamma_j(z))$	`be[i,j,z], ga[i,j,z]`
bosonic current $i\partial^{i+1}\phi_j(z)$	`pj[i,j,z]`

In what follows, we write the free fields as f. These fields satisfy

$$\begin{aligned}
\psi_i(z)\psi_j(w) =&\frac{\delta_{ij}}{z-w}, \quad b_i(z)c_j(w) = \frac{\delta_{ij}}{z-w}, \quad c_i(z)b_j(w) = \frac{\delta_{ij}}{z-w}, \\
\beta_i(z)\gamma_j(w) =&\frac{\delta_{ij}}{z-w}, \quad \gamma_i(z)\beta_j(w) = -\frac{\delta_{ij}}{z-w}, \quad i\partial\phi_i(z)i\partial\phi_j(w) = \frac{\delta_{ij}}{(z-w)^2}.
\end{aligned}$$

The indices of the above free fields can be any values and any variables that *Mathematica* allows. However, we recommend the use of the sequential positive integers beginning from 1 in order to utilize the functions in the package. If they are variables, then the others than the OPE of free fields of the same symbols vanish in this package, which means that

$$\mathtt{ope[ps[0,i,a],ps[0,j,z]]} == \mathtt{0}, \quad \mathtt{ope[ps[0,j,a],ps[0,j,z]]} == \mathtt{1/(a-z)}.$$

We mean by "`==`" that the equation is correct. The function `ope[]` is explained in sec.3.1..

2.2. Vertex operators

vertex operator : $\partial^j e^{i(q_1\phi_1+\cdots q_N\phi_N)}: c(\{q_1,\cdots,q_N\},\{\hat{p}_1,\cdots,\hat{p}_N\})$
`vp[j,{q1,q2,...,qN},z]`

Let ϕ, $\mathtt{q}$ and $\hat{p}$ denote the vectors $\{\phi_1,\cdots,\phi_N\}$, $\{\mathtt{q}_1,\cdots,\mathtt{q}_N\}$ and $\{\hat{\mathtt{p}}_1,\cdots,\hat{\mathtt{p}}_N\}$. We also use the vertex operator $: e^{i\mathtt{q}\cdot\phi} : c(\mathtt{q},\hat{\mathtt{p}}) =: \sum_{n=0}^{\infty}\frac{(i\mathtt{q}\cdot\phi)^n}{n!} : c(\mathtt{q},\hat{\mathtt{p}})$ which is the normal ordered operator of bosons ϕ. We see that the field `vp[j,q,z]` containes not only $: e^{i\mathtt{q}\cdot\phi} :$ but also the cocycle factor $c(\mathtt{q},\hat{\mathtt{p}})$, where $\hat{\mathtt{p}}_j$ measures the eigen value of the zero mode of the bosonic current $i\partial\phi_j =$ `pj[0,j,z]`. We ignore the explanation of the factor in this section.

A regular expression, which now we define, of the vertex operator `vp[j,q,z]` is not $: \partial^j e^{i\mathtt{q}\cdot\phi} : c(\mathtt{q},\hat{\mathtt{p}})$ but $: \mathcal{P}_j e^{i\mathtt{q}\cdot\phi} : c(\mathtt{q},\hat{\mathtt{p}})$, where $\mathcal{P}_j$ is a differential polynomial of the free fields $\phi_1,\cdots,\phi_N$. Every output is represented in the regular expression. The function `Fsort[]` gives the regular form of operators.

The vertex operator `vp[j,{q1,q2,···,qN},z]` has $u(1)$ charges of the bosons ϕ_1, ϕ_2, ..., ϕ_N, which is represented as

$$\mathtt{ope[pj[0,1,z],vp[0,\{q_1,q_2,\cdots,q_N\},w]]} == \frac{\mathtt{q_1}}{\mathtt{z-w}}\mathtt{vp[0,\{q_1,q_2,\cdots,q_N\},w]}.$$

Therefore, we have to use the only positive integers from 1 to N for the indicies of the bosons `pj[]`, when we treats the vertex operator.

The OPE between vertex operators with parameters `p={p1,···,pN}` and `q={q1,···,qN}` will give a form

$$\mathtt{ope[vp[0,p,a],vp[0,q,z]]} == \mathtt{(a-z)\hat{}(p.q)N1[vp[0,p,a],vp[0,q,z]]},$$

where `(p.q)`$(=\sum_{i=1,N}\mathtt{p}_i\mathtt{q}_i)$ is the inner product, and the vertex operator `vp[0,p,a]` is not expanded around the position `z`, and `N1[]` is the function representing the normal ordered product of operators, which is expalined in the next section. If the value of $(p.q)$ is evaluated, `ope[vp[0,p,a],vp[0,q,z]]` will give a expanded form. If one wants to know the coefficient of `(a-z)^(p.q+1)`, it is better to use the function `res[]` as

$$\mathtt{res[p.q+1,vp[0,p,a],vp[0,q,z]]} == \mathtt{CO2[p,q]\,Sum[}$$
$$\mathtt{N1[p[[i]]\,pj[0,i,z],vp[0,p+q,z]],\{i,1,Length[p]\}]},$$

where `CO2[p,q]` is the cocycle factor $c(p,q)$, which is the c-number function discussed in sec.4. In what follows, we write the vertex operator with the charge vector q as V_q.

2.3. Normal Ordered Operators

To represent the normal ordered product of operators, we use the function `Nl[]`. `Nl[`$f_1, f_2, \cdots, f_n, V$`]` means the normal ordered operator of the fields f_1, $f_2, \cdots, f_n$ and the vertex V, which we call a monomial with the vertex operators. `Nl[`$f_1, f_2, \cdots, f_n$`]` means the normal ordered operator of the fields f_1, $f_2, \cdots$, f_n, which we call a monomial. An operators *opr* is the free filed itself, or the monomial, or the polynomial, which is the sum of monomials. Then, `Nl[`$opr_1, opr_2, \cdots, opr_n$`]` is also a normal ordered operator.

Since the normal ordered product of vertex operator `Nl[`V_q`,`V_p`]` is equivalent to the vertex operator `CO2[q,p]` V_{q+p}, it is enough to represent operator monomials including vertex operator by a single vertex operator.

Let us summarize the "operator" representing the operator polynomial. We write such an operator as *opr*.

normal ordered operator monomials

$$opr = \texttt{Nl[}f_1, f_2, ..., f_n\texttt{]}\ ,\ \texttt{Nl[}f_1, f_2, ..., f_n, V_q\texttt{]}\ ,\ f\ ,\ V_q$$

normal ordering of operators $\qquad opr' = \texttt{Nl[}\ldots, opr, \ldots\texttt{]}$

a linear combination with c-number coefficients c_i

$$opr = c_1\, opr_1 + c_2\, opr_2 + ...$$

The function `Nl[]` automatically reduces to the simplest form using the following process

$$\begin{aligned}
\texttt{Nl[..., a + b , ...]} &\rightarrow \texttt{Nl[..., a, ...]} + \texttt{Nl[..., b, ...]},\\
\texttt{Nl[Nl[...]]} &\rightarrow \texttt{Nl[...]},\\
\texttt{Nl[...,}c_1\, opr\texttt{, ...]} &\rightarrow c_1\ \texttt{Nl[...,}opr\texttt{, ...]}.
\end{aligned}$$

Note that the normal order manipulation `Nl[`opr_1, opr_2`]` does not mean the regular terms of the operator product of opr_1 and opr_2.

As the simplest example for operator polynomials, we give the case of the Wakimoto realization of $sl(2)$ [3]. We use a set of (β, γ) and $i\partial\phi$. The three currents are expressed as

```
JE[z_] :=  be[0,1,z];
JH[z_] :=  Nl[be[0,1,z],ga[0,1,z]] + h[1] pj[0,1,z];
JF[z_] :=  f[1] ga[1,1,z] + f[2] Nl[ga[0,1,z],pj[0,1,z]]+
           Nl[be[0,1,z],ga[0,1,z],ga[0,1,z]];
```

If `f[1] = 2 (1- h[1]`^{2}`)` and `f[2] = 2 h[1]`, then these satisfy the level `k=1- h[1]`^{2} $sl(2)$ Kac-Moody algebra.

2.4. Stress tensors and $u(1)$ currents

stress tensor for ψ_i	`Tps[i,z]`
stress tensor for (b_i, c_i) s.t. the conformal weight of b_i is h	`Tbc[h,i,z]`
stress tensor for (β_i, γ_i) s.t. the conformal weight of β_i is h	`Tbg[h,i,z]`
stress tensor for $i\partial\phi_i$ s.t. the anomaly of $i\partial\phi_i$ is a	`Tpj[a,i,z]`
fermion number current for (b_i, c_i) normalized to be 1 for b_i	`Jc[i,z]`
boson number current for (β_i, γ_i) normalized to be 1 for β_i	`Jg[i,z]`
bosonic current $i\partial\phi_i$	`Jp[i,z]`

For convenience, stress tensors and $u(1)$ currents for all free fields are prepared in "ope.math". Producting the stress tensor T on the operator φ, we have the OPE

$$T(z)\varphi(w) = \cdots + \frac{h_\varphi \varphi(w)}{(z-w)^2} + \frac{\partial\varphi(w)}{z-w} + \textit{regular terms},$$

then the constant h_φ is called the conformal weight of φ. By definition, a primary field produce at most the second pole under the OPE with the stress tensor. All the free fields $\psi_i, b_i, c_i, \beta_i, \gamma_i$ (but $i\partial\phi_i$) are primary. For $i\partial\phi_i$, it may have an anomaly, if the stress tensor has the linear term of the scalar field.

$$T(z)i\partial\phi(w) = \frac{a}{(z-w)^3} + \frac{i\partial\phi(w)}{(z-w)^2} + \frac{i\partial^2\phi(w)}{z-w} + \textit{regular terms},$$

where the residue of the third pole term is called the anomaly.

The $u(1)$ current `Jc[i,z]=` $b_i c_i$ (`Jg[i,z]=` $-\beta_i\gamma_i$) counts the fermion (boson) number as

$$\texttt{ope[Jc[i,a],bi[0,i,z]]==}\ \frac{\texttt{bi[0,i,z]}}{\texttt{a-z}}, \quad \texttt{ope[Jc[i,a],ci[0,i,z]]==} - \frac{\texttt{ci[0,i,z]}}{\texttt{a-z}},$$
$$\texttt{ope[Jg[i,a],be[0,i,z]]==}\ \frac{\texttt{be[0,i,z]}}{\texttt{a-z}}, \quad \texttt{ope[Jg[i,a],ga[0,i,z]]==} - \frac{\texttt{ga[0,i,z]}}{\texttt{a-z}}.$$

The $u(1)$ current `Jp[i,z]=` $i\partial\phi_i(z)$ generates translation of $\phi_i(z) = -$`I pj[-1,i,z]` as

$$\texttt{ope[Jp[i,a],pj[-1,i,z]] ==}\frac{\texttt{1}}{\texttt{a-z}}.$$

2.5. To construct currents with coefficients

reconstruct a current with coefficient `Cname` from *opr*
`Current[`*opr*`,z,Cname]`

The function `Current[P(a),z,Cname]` reconstructs a polynomial with the coefficients named `Cname` and the basis which are the monomials appearing in the polynomial `P`. Let x_i's be the basis of the polynomial `P`, then the function maps

$$\mathtt{P}(z_1) \to \sum_i \mathtt{Cname[i]} x_i(z).$$

This function `Current[]` is applicable for searching free field realization of the current algebras. When we look for the free field representation of algebras, we make a differential polynomial with some coefficients undetermined, and then we determine the coefficients by requiring the algebra. A differential polynomial of free fields quickly becomes long, if we treat many free fields and we construct higher spin operators.

W_3 *algebra.* Let us construct W_3 algebra in terms of two bosons. Using a formal $u(1)$ current `J[z_]:= Jp[1,z]+Jp[2,z];`, we define the spin 3 current as

$$\mathtt{W[z\_]} := \mathtt{Current[Nl[J[z],J[z],J[z]] + Nl[J[z],der[J[z],z]] + der[2,J[z],z],z,f]};$$

which produces a polynomial of ten terms long. Requiring W_3-algebra

$$\begin{aligned} T(z)T(w) = \quad & \tfrac{c}{2(z-w)^4} + \tfrac{2}{(z-w)^2}T(w) + \tfrac{\partial T(w)}{z-w}, \\ T(z)W(w) = \quad & \tfrac{3}{(z-w)^2}W(w) + \tfrac{\partial W(w)}{z-w}, \\ W(z)W(w) = \quad & \tfrac{c_6}{(z-w)^6} + \tfrac{c_4}{(z-w)^4}T(w) + \tfrac{c_3}{(z-w)^3}\partial T(w) + \\ & \tfrac{c_2 \mathit{reg}(TT)(w) + \tilde{c}_2\partial^2 T(w)}{(z-w)^2} + \tfrac{c_1\partial \mathit{reg}(TT)(w) + \tilde{c}_1\partial^3 T(w)}{z-w}, \end{aligned}$$

we can find solutions for the coefficients `f[]`. The stress tensor T is given in the OPE $W(a)W(z)$ as `res[-4,W[a],W[z]]==`c_4 `T[z]`. The operator $\mathit{reg}(TT)$ is the regular term in the OPE of the stress tensor T and itself, which is given by `res[0,T[a],T[z]]`. One of the solutions is

$$\begin{aligned} W = \quad & \tfrac{\sqrt{3}h}{2} i\partial^2\phi_1 i\partial\phi_1 + hi\partial^2\phi_1 i\partial\phi_2 - \tfrac{\sqrt{3}h}{2} i\partial^2\phi_2 i\partial\phi_2 - (i\partial\phi_1)^2 i\partial\phi_2 + \\ & \tfrac{1}{3}(i\partial\phi_2)^3 - \tfrac{\sqrt{3}h^2}{4} i\partial^3\phi_1 + \tfrac{h^2}{4} i\partial^3\phi_2, \\ T = \quad & \tfrac{1}{2}(i\partial\phi_1)^2 + \tfrac{1}{2}(i\partial\phi_2)^2 - \tfrac{h}{2} i\partial^2\phi_1 - \tfrac{\sqrt{3}h}{2} i\partial^2\phi_2, \end{aligned}$$

and $c_6 = \frac{(8-15h^2)(1-6h^2)}{3}$, $c_4 = 8 - 15h^2$, $c_3 = \frac{8-15h^2}{2}$, $c_2 = 4$, $\tilde{c}_2 = -\frac{9h^2}{4}$, $c_1 = 2$, $\tilde{c}_1 = -\frac{2+3h^2}{6}$ and $c = 2(1 - 6h^2)$.

In the above example, using the current J is enough to construct W. When we construct charged currents, we need to know how many charged operators equivalent to currents modulo J exist in the free field realization. The Kac-Moody algebra is one of such situations. In constructing Kac-Moody algebra, we first construct currents of the nilpotent subalgebra corresponding to the positive roots and the lowest root current. Then, the other negative root currents and the null root current are obtained through the algebra.

We give $sl(3)$ Kac-Moody algebra for the case. We use three β_i, γ_i, $i = 1, 2, 3$ for the three positive roots α_1, α_2 and $\alpha_1 + \alpha_2$ of $sl(3)$ and two $i\partial\phi_i$, $i = 1, 2$ for the two Cartan subalgebras. In this case, J is defined by $\mathtt{J[a\_]} := \mathtt{Jg[1,a]}+\mathtt{Jg[2,a]}+\mathtt{Jg[3,a]}+\mathtt{Jp[1,a]}+\mathtt{Jp[2,a]}$. The sufficient sets are $\{\beta_1, \beta_3\gamma_2\}$ for α_1, $\{\beta_3\}$ for $\alpha_1 + \alpha_2$, $\{\beta_2, \beta_3\gamma_1\}$ for α_2, $\{\gamma_1, \beta_2\gamma_3\}$ for $-\alpha_1$, $\{\gamma_3, \gamma_2\gamma_1\}$ for $-\alpha_1 - \alpha_2$ and $\{\gamma_2, \beta_1\gamma_3\}$ for $-\alpha_2$. For the null roots, though there is another composite operator than `Jg[i,z]=`$-\beta_i\gamma_i$ with vanishing charges and the conformal weight 1, that is $\beta_3\gamma_1\gamma_2$, it can not be a $u(1)$ current. Define the currents as

$$
\begin{aligned}
\mathtt{JE1[z\_]} :=&\quad \mathtt{Current[be[0,1,a] + Nl[be[0,3,a], ga[0,2,a]], z, e1]};\\
\mathtt{JE2[z\_]} :=&\quad \mathtt{Current[be[0,2,a] + Nl[be[0,3,a], ga[0,1,a]], z, e2]};\\
\mathtt{JE3[z\_]} :=&\quad \mathtt{be[0,3,z]};\\
\mathtt{JLowest[z\_]} :=&\quad \mathtt{ga[0,3,z] + Nl[ga[0,1,z], ga[0,2,z]]};\\
\mathtt{JF3[z\_]} :=&\quad \mathtt{Current[Nl[J[a], JLowest[a]] + der[JLowest[a], a], z, h]};\\
\mathtt{JH1[z\_]} :=&\quad \mathtt{JH1[z] = Module[a, res[-1, JE1[a], JF1[z]]]};\\
\mathtt{JH2[z\_]} :=&\quad \mathtt{JH2[z] = Module[a, res[-1, JE2[a], JF2[z]]]};\\
\mathtt{JF1[z\_]} :=&\quad \mathtt{JF1[z] = Module[a, res[-1, JE2[a], JF3[z]]]};\\
\mathtt{JF2[z\_]} :=&\quad \mathtt{JF2[z] = Module[a, res[-1, JE1[a], JF3[z]]]};
\end{aligned}
$$

where `Jlowest[]` is defined as a polynomial to have the same charges as γ_3, and the functions `res[]` and `der[]` in the above are explained in sec.3.2., and `Module[]` is the function prepared in the *Mathematica*. To simplify the structure of the realization, we can set the coefficient `e2[2] = 0`, thus we will start with $\mathtt{JE2[z]} = \mathtt{be[0,2,z]}$. Requiring the $sl(3)$ Kac-Moody algebra, we can find a solution

$$
\begin{aligned}
E_1 =&\quad \beta_1 + \gamma_2\beta_3, \qquad E_2 = \beta_2, \qquad E_3 = \beta_3,\\
F_3 =&\quad \tfrac{6-h^2}{2}\partial\gamma_3 - \tfrac{h}{2}\gamma_3 i\partial\phi_1 + \tfrac{\sqrt{3}h}{2}\gamma_3 i\partial\phi_2 + \tfrac{-4+h^2}{2}\partial\gamma_1\gamma_2 +\\
&\quad \beta_1\gamma_1\gamma_3 + \beta_2\gamma_2\gamma_3 + \beta_3\gamma_3^2 + h\gamma_1\gamma_2 i\partial\phi_1 - \beta_1\gamma_1^2\gamma_2.
\end{aligned}
$$

Other currents are obtained through OPEs as

$$\begin{aligned}
H_1 = &\quad -2\beta_1\gamma_1 + \beta_2\gamma_2 - \beta_3\gamma_3 + hi\partial\phi_1, \\
H_2 = &\quad -\beta_1\gamma_1 + 2\beta_2\gamma_2 + \beta_3\gamma_3 + \tfrac{h}{2}i\partial\phi_1 + \tfrac{\sqrt{3}h}{2}i\partial\phi_2, \\
F_1 = &\quad \tfrac{-4+h^2}{2}\partial\gamma_1 + \beta_2\gamma_3 + h\gamma_1 i\partial\phi_1 - \beta_1\gamma_1^2, \\
F_2 = &\quad \tfrac{6-h^2}{2}\partial\gamma_2 + \beta_1\gamma_3 + \tfrac{h}{2}\gamma_2 i\partial\phi_1 + \tfrac{\sqrt{3}h}{2}\gamma_2 i\partial\phi_2 - \beta_1\gamma_1\gamma_2 + \beta_2\gamma_2^2 + \beta_3\gamma_3\gamma_2.
\end{aligned}$$

The level of the algebra is $k = -6 + h^2$.

2.6. Local parameters

Local parameters are commuting or anti-commuting fields.

j-th bosonic local parameter	`xi[`*derivative*`,j,`*position*`]`
j-th fermionic local parameter	`ro[`*derivative*`,j,`*position*`]`

Local parameters should be used in the function `Nl[]` just as if they are free fields.

<u>*example:*</u>

$$\mathtt{res}[-1, \mathtt{Nl}[\mathtt{Tpj}[\mathtt{h},1,\mathtt{a}], \mathtt{xi}[0,1,\mathtt{a}]], \mathtt{pj}[-1,1,\mathtt{z}]] == \mathtt{h} * \mathtt{xi}[1,1,\mathtt{z}]/2 + \mathtt{Nl}[\mathtt{xi}[0,1,\mathtt{z}], \mathtt{pj}[0,1,\mathtt{z}]],$$

where `pj[-1,1,z]` denotes $i\phi$. This means the variation

$$\delta_\xi \phi(z) \equiv \int \frac{da}{2\pi i} T(a)\xi(a)\phi(z) = \frac{h}{2}\partial\xi(z) + \xi(z)\partial\phi(z).$$

2.7. Super field formalism

The super field formalism is useful to construct currents in an explicitly supersymmetric form. We give a simple example to construct $N=1$ super stress tensor in terms of a single real boson $\phi(z) = -$`I pj[-1,1,z]` and a single Majorana fermion $\psi =$`ps[0,1,z]`. Let θ be a Grassman odd variable, and (z,θ) be the coordinate in the superspace, and $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ be a super covariant derivative, and $\Phi = \phi + \theta\psi$ be a super field. Then, the super stress tensor is represented as $S(\theta, z) = \frac{1}{2}D\Phi\partial\Phi + h\partial D\Phi$.

In order to represent the variable θ, we use a local parameter `ro[]` and set it independent on the bosonic coordinate z by the condition

$$\texttt{ro}[\texttt{i\_},\texttt{j\_},\texttt{z\_}] := \texttt{0}/;\texttt{i} > \texttt{0};$$

Preparing $D\Phi$ and $DD\Phi = \partial\Phi$ as

$$\begin{aligned}\texttt{DP}[\texttt{i\_},\texttt{z\_}] :=&\quad \texttt{ps}[0,1,\texttt{z}] - \texttt{I}\,\texttt{Nl}[\texttt{ro}[0,\texttt{i},\texttt{z}],\texttt{pj}[0,1,\texttt{z}]];\\ \texttt{DDP}[\texttt{i\_},\texttt{z\_}] :=&\quad -\texttt{I}\,\texttt{pj}[0,1,\texttt{z}] + \texttt{Nl}[\texttt{ro}[0,\texttt{i},\texttt{z}],\texttt{ps}[1,1,\texttt{z}]];\end{aligned}$$

we can construct the super stress tensor as

$$\texttt{S}[\texttt{i\_},\texttt{z\_}] := \texttt{Nl}[\texttt{DP}[\texttt{i},\texttt{z}],\texttt{DDP}[\texttt{i},\texttt{z}]]/2 + \texttt{h}\,\texttt{der}[\texttt{DP}[\texttt{i},\texttt{z}],\texttt{z}];$$

Then, we can compute the OPE of $S(\theta_1, z)$ and $S(\theta_2, w)$ by `ope[S[1,z],S[2,w]]`.

3. Operator product expansions

We will explain the functions which map operator polynomials to operator polynomials supplied in “ope.math”.

3.1. OPE

operator product expansion	`ope[`opr_1, opr_2`]`
operator product expansion with weight `p`	`ope[p,`opr_1, opr_2`]`

Singular terms of an OPE of operators $opr_1(z)$ and $opr_2(w)$ are evaluated by the function `ope[`$opr_1(z)$`,`$opr_2(w)$`]`.

$$\texttt{ope[}opr_1(z)\texttt{,}opr_2(w)\texttt{]} = \text{ Singular terms of } opr_1(z)opr_2(w).$$

The function `ope[`$p, opr_1(z), opr_2(w)$`]` gives

$$\texttt{ope[p,}opr_1(z), opr_2(w)\texttt{]} = \text{ Singular terms of } (z-w)^{\texttt{p}} opr_1(z)opr_2(w),$$

where `p` can be any c-number function. If `p` has really the difference of an integral number from the power of $(z - w)$ the OPE of $opr_1(z)$ $opr_2(w)$, then `ope[p,`$opr_1(z), opr_2(w)$`]` gives an expanded form. For example, let us consider the OPE

$$b_1(z)c_1(w) = \frac{1}{z-w} + b_1c_1(w) + (z-w)\partial b_1c_1(w) + \cdots,$$

which is represented by the function `ope[]` as

$$\texttt{ope[bi[0,1,z],ci[0,1,w]]} == \texttt{1/(z-w)}.$$

Then, if we want to show $(z-w)\partial b_1c_1(w)$ term, we may use the function `ope[]` as

$$\texttt{ope[-2,bi[0,1,z],ci[0,1,w]]==} \quad \frac{1}{(z-w)^3} + \frac{1}{(z-w)^2}\texttt{Nl[bi[0,1,w],ci[0,1,w]]}+ \frac{1}{z-w}\texttt{Nl[bi[1,1,w],ci[0,1,w]]}.$$

For the vertex operators $\texttt{vp[0,\{q\},z]}$ and $\texttt{vp[0,\{p\},z]}$, we have the OPE

$$\texttt{ope[vp[0,\{q\},a],vp[0,\{p\},z]]} == \texttt{(a-z)}^{\texttt{qp}}\texttt{Nl[vp[0,\{q\},a],vp[0,\{p\},z]]},$$

as we have explained before in sec.2.3.. If we want to show the $(a-z)^{\texttt{qp}+1}$ term, we may use the function `ope[]` as

$$\texttt{ope[-qp-1,vp[0,\{q\},a],vp[0,\{p\},z]]} == \texttt{Nl[vp[0,\{q+p\},z]]/(a-z)}.$$

The OPE of operators whose positions are the same is regarded as a normal ordered product. Then `ope[`$opr_1(z)$`,`$opr_2(z)$`]`$= 0$.

We briefly explain the procedure to compute OPEs in the package. First, the OPE of polynomials is separated to the OPEs of each monomials.

$$\texttt{ope}[c_1 opr_1 + \cdots, c_1' opr_1' + \cdots] \to c_1c_1'\texttt{ope}[opr_1, opr_1'] + \cdots$$

The monomials are represented as `Nl[`f_1`,`f_2`,`$\cdots$`,`f_m`,`V`]`, where f_i is the free field and V is a vertex operator that is explained in the next section. The OPE of monomials are computed automatically as follows: First, the vertex operator part is computed as

$$\begin{aligned}
\texttt{ope}[\texttt{Nl}[f_1,f_2,\cdots,f_m,V],\ & \texttt{Nl}[f_1',f_2',\cdots,f_n',V']] \\
\to\ & \texttt{Monomial}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\texttt{Nl}[f_1',f_2',\cdots,f_n'],\texttt{VNl}[\texttt{Vev}[V,V']]] \\
\texttt{ope}[\texttt{Nl}[f_1,f_2,\cdots,f_m,V],\ & \texttt{Nl}[f_1',f_2',\cdots,f_n']] \\
\to\ & \texttt{Monomial}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\texttt{Nl}[f_1',f_2',\cdots,f_n'],\texttt{VNl}[\texttt{Nl}[V]]] \\
\texttt{ope}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\ & \texttt{Nl}[f_1',f_2',\cdots,f_n',V']] \\
\to\ & \texttt{Monomial}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\texttt{Nl}[f_1',f_2',\cdots,f_n'],\texttt{VNl}[\texttt{Nl}[V']]] \\
\texttt{ope}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\ & \texttt{Nl}[f_1',f_2',\cdots,f_n']] \\
\to\ & \texttt{Monomial}[\texttt{Nl}[f_1,f_2,\cdots,f_m],\texttt{Nl}[f_1',f_2',\cdots,f_n'],\texttt{VNl}[]]
\end{aligned}$$

where the functions `Monomial[]`, `VNl[]` and `Vev[]` are the functions used inside of the package, which represent "monomial", "normal ordered operator containing the vertex operator", by which we mean that there is no vertex operator in the other funtions `Nl[]` than in the `VNl[]` in the function `Monomial[]`, and "vacuum expectation", respectively. The function `Vev[]` is evaluated by the elementary data for free fields and vertex operators, which is given in the sec.2.1. and the sec.2.2., respectively.

Second, all free fields in the function `Nl[]` are recursively moved into `VNl[]` as

$$\begin{aligned}
&\texttt{Monomial}[\texttt{Nl}[\cdots],\texttt{Nl}[f_1,f_2,\cdots,f_m],\texttt{VNl}[\texttt{Nl}[f'_1,f'_2,\cdots,f'_n,V']]]\\
\rightarrow\quad &\texttt{Monomial}[\texttt{Nl}[\cdots],\texttt{Nl}[f_1,f_2,\cdots,f_{m-1}],\texttt{VNl}[\texttt{Nl}[f_m,f'_1,f'_2,\cdots,f'_n,V']]]+\\
&\textstyle\sum_{i=1,n}\texttt{Monomial}[\texttt{Nl}[\cdots],\texttt{Nl}[f_1,f_2,\cdots,f_{m-1}],\texttt{VNl}[\texttt{Nl}[\cdots,\texttt{Vev}[f_m,f'_i],\cdots,V']]]+\\
&\texttt{Monomial}[\texttt{Nl}[\cdots],\texttt{Nl}[f_1,f_2,\cdots,f_{m-1}],\texttt{VNl}[\texttt{Nl}[f'_1,f'_2,\cdots,f'_n,\texttt{Vev}[f_m,V']]]].
\end{aligned}$$

Finally, we have monomials of form `Monomial[VNl[Nl[···]]]`, which is represented in the output as `Nl[···]`.

In the above explanation, we do not give exactly the same form for the inside functions and omit the consideration of the statistics of fields for simplicity. If you want to know the exact form, you can see it in the source of the package.

3.2. Other Manipulations

coefficients of `p`-th zeros ((−`p`)-th poles) of OPE	`res[p,`opr_1`,`opr_2`]`
`i`-th derivative	`der[i,`opr`,z]`, `der[`opr`,z]` (= `der[1,`opr`,z]`)
sorting free fields in $opr(z)$	`Fsort[`opr`,z]`
expansion of $(\mathtt{z}-\mathtt{w})^{\mathtt{p}}opr(z)$ at w	`Texp[p,z,w,`opr`]`

The function `res[p,`opr_1`,`opr_2`]` gives the coefficient of the `p`-th zero (or (−`p`)-th pole) of the OPE of the currents opr_1 and opr_2. As mentioned before in sec.2.2., a regular part of the OPE of opr_1 and opr_2 is given not by `Nl[`opr_1`,`opr_2`]` but by `res[0,`opr_1`,`opr_2`]`. We note that `res[0,`opr_1`,`opr_2`]` is not always the same as `res[0,`opr_2`,`opr_1`]`.

The function `der[i,`opr`,z]` gives the `i`-th derivative of the current opr at `z`. We should note that it is not allowed in the package to use the function `D[`opr`,z]`, which is the function of the derivative prepared in the *Mathematica*, to get derivatives of operators. This is because the functions defined in the package "ope.math" do not know hot to treat `D[`opr`,z]`'s.

The function `Fsort[`opr`,z]` gives a sorted expression for the current opr at `z` *w.r.t.* a rule for the order of free fields which we have fixed in the package.

The function Texp[p,z,w,opr] gives the singular part of the expansion of $(z-w)^p opr$ at w. As an example, we show

$$\texttt{Texp}[-2,\texttt{z},\texttt{w},\texttt{bi}[0,1,\texttt{a}]] == \frac{\texttt{bi}[0,1,\texttt{w}]}{(\texttt{z}-\texttt{w})^2}+\frac{\texttt{bi}[1,1,\texttt{w}]}{\texttt{z}-\texttt{w}},$$

where the position a in bi[0,1,a] is a dummy variable.

3.3. Mode expansion from an OPE

If one wants to extract information about the algebras among the modes of the generators and free fields, one needs only to use res[]. It is better to explain how we could use res[] by taking the example of a Virasoro algebra. Let T[z]=Tpj[h,1,z] be a Virasoro algebra with the central charge c=1-3h^2, and P[k,z]=vp[0,{k},z] (vp[] is a vertex operator explained in the next section.) be a primary field with the conformal weight Δ=k(h+k)/2. T[z] is expanded as T[z]=$\sum_{\texttt{n}} \texttt{T}_{\texttt{n}}\texttt{z}^{-\texttt{n}-2}$. We can define a procedure Lmod[] as

$$\texttt{Lmod}[\{\texttt{i1\_},\texttt{i2\_\_}\},\texttt{p\_}] := \texttt{Lmod}[\{\texttt{i1},\texttt{i2}\},\texttt{p}] = \texttt{Module}[\{\texttt{a}\},\texttt{res}[-2-\texttt{i1},\texttt{T}[\texttt{a}],\texttt{Lmod}[\{\texttt{i2}\},\texttt{p}]]];$$
$$\texttt{Lmod}[\{\texttt{i\_}\},\texttt{p\_}] := \texttt{Lmod}[\{\texttt{i}\},\texttt{p}] = \texttt{Module}[\{\texttt{a}\},\texttt{res}[-2-\texttt{i},\texttt{T}[\texttt{a}],\texttt{p}]];$$

by which we can obtain $\texttt{T}_{\texttt{i1}}\texttt{T}_{\texttt{i2}}...|p\rangle$. To demonstrate how we could obtain the null vector, we consider c= 1/2 (Ising) case (h= $1/\sqrt{6}$). For descendants of the primary P[m/$\sqrt{6}$,z], we search for the null state condition. For the level one state P1= $\texttt{T}_{-1}$P[m/$\sqrt{6}$,z]

$$\texttt{P1= Lmod}[\{-1\},\texttt{P}[\texttt{m}/\sqrt{6},\texttt{z}]];$$

if we set $\texttt{T}_1$ P1 = $\texttt{T}_2$ P1=0, then we have m=0,-1. These two solutions mean that P is a primary state with $\Delta=0$. For the level two state P2=($\texttt{T}_{-2}$+a $\texttt{T}_{-1}\texttt{T}_{-1}$)P[m/$\sqrt{6}$,z]

$$\texttt{P2=Lmod}[\{\texttt{-2}\},\texttt{P}[\texttt{m}/\sqrt{6},\texttt{z}]]\texttt{+A Lmod}[\{\texttt{-1,-1}\},\texttt{P}[\texttt{m}/\sqrt{6},\texttt{z}]];$$

if we set $\texttt{T}_1$ P2 = $\texttt{T}_2$ P2=0, then we obtain m=2,-3,1/2,-3/2 and A=-4/3,-4/3,-3/4,-3/4, respectively. These four solutions mean that P is a primary state with $\Delta=1/2$ or $\Delta=1/16$.

4. Vertex operators and cocycle factors

cocycle factor	`CO2[{p1,p2,...},{q1,q2,...}]`

The vertex operator `vp[0,` a, z`]`, which is already introduced in sec.2.3., means the form $v(a,z) =: \exp(ia\cdot\phi(z)) : c(a,\hat{p})$, where $: \exp(ia\cdot\phi(z)) :$ means the normal ordering with respect to the boson oscillator, and $\hat{p}$ counts $u(1)$ charges of bosons, and $c(a,\hat{p})$ is the cocycle factor we will now explain.

The OPE between this kind of operators is computed as

$$v(a,z)v(b,w) = (z-w)^{a\cdot b} : \exp(ia\cdot\phi(z))\exp(ib\cdot\phi(w)) : c(a,b+\hat{p})c(b,\hat{p}). \tag{4.1}$$

If we set $b=-a$, then the R.H.S. of the Eq.(4.1) is not a vertex operator, and we should require

$$c(a,-a+\hat{p})c(-a,\hat{p}) = 1$$

for any value $\hat{p}$. Therefore, we know that $v(a,z)$ is a fermion (boson) if $(a\cdot a)$ is odd (even). It is easy to consider only the cases where every vertex operator is bosonic or fermionic, because then all the cocycle factors can be 1 or -1.

Requiring that the R.H.S. of the Eq.(4.1) is a vertex operator $v(a+b,w)$, the relation

$$c(a,b+\hat{p})c(b,\hat{p}) = c(a,b)c(a+b,\hat{p})$$

should be satisfied. Now we can rewrite the OPE as

$$v(a,z)v(b,w) = (z-w)^{a\cdot b}c(a,b)v(a,z|b,w),$$

where $v(a,z|b,w) =: \exp(ia\cdot\phi(z))\exp(ib\cdot\phi(w)) : c(a+b,\hat{p})$. To change OPEs into (anti-)commutation relations, we need to compute

$$\begin{aligned}&v(a,m)v(b,n) - (-)^{(a\cdot a)(b\cdot b)}v(b,n)v(a,m) = \\ &\oint_{w,0}\frac{dzz^m}{2\pi i}\oint_0\frac{dww^n}{2\pi i}v(a,z)v(b,w) - (-)^{(a\cdot a)(b\cdot b)}\oint_{z,0}\frac{dww^n}{2\pi i}\oint_0\frac{dzz^m}{2\pi i}v(b,w)v(a,z).\end{aligned} \tag{4.2}$$

If the coefficient $c(b,a)$ satisfies

$$c(b,a) = \exp(i\pi(-(a\cdot a)(b\cdot b) - (a\cdot b)))\,c(a,b),$$

it follows that

$$\text{R.H.S. of (4.2)} = c(a,b) \oint_w \frac{dz z^m}{2\pi i} \oint_0 \frac{dw w^n}{2\pi i} (z-w)^{a\cdot b} v(a,z|b,w).$$

Let us summarize the properties of the factor $c(a,b)$. It should satisfy

$$\begin{aligned} &(i) \quad c(a,-a+b)c(-a,b) = 1, \\ &(ii) \quad c(a,b+d)c(b,d) = c(a,b)c(a+b,d), \\ &(iii) \quad c(b,a) = \exp\left(i\pi(-(a\cdot a)(b\cdot b) - (a\cdot b))\right) c(a,b), \end{aligned}$$

and furthermore we can set

$$(iv) \quad c(a,0) = 1.$$

The condition (iii) is required only for $a^2, b^2, a\cdot b \in \mathbf{Z}$ which we call a pair of integral points a and b. To introduce this factor in the "ope.math", we define the following manipulations:

$$\begin{aligned} &(1) \quad \mathtt{Nl[vp[0,a,z],vp[0,b,z]] := CO2[a,b]Nl[vp[0,a+b,z]];} \\ &(2) \quad \mathtt{Nl[vp[0,a,z],vp[0,-a,z]] := 1;} \\ &(3) \quad \mathtt{CO2[b,a] := Exp[I\,Pi(-(a.a)(b.b)-(a.b))]CO2[a,b]/;} \\ &\qquad \mathtt{b < a\&\&mod[a\hat{}2,1] == 0\&\&mod[b\hat{}2,1] == 0\&\&mod[a.b,1] == 0;} \end{aligned}$$

The condition $b < a$ is examined by using the lexicographic order. It is necessary to evaluate $c(a,b)$ correctly. As seen in the Frenkel-Kac construction [8] of simply-laced level one Kac-Moody algebras, $c(a,b)$ itself represents the structure constant of the algebra.

4.1. Spin operators for a complex fermion

A complex fermion can be represented as a vertex operator as $b = \exp(i\phi)$ and $c = \exp(-i\phi)$. Spin operators are also constructed as $S_+ = \exp(i\phi/2)$ and $S_- = \exp(-i\phi/2)$. We define the operators as

$$\mathtt{S[i\_,z\_] := vp[0,\{i/2\},z]; b[z\_] := S[2,z]; c[z\_] := S[-2,z];}$$

We compute OPEs among these operators,

$$\begin{aligned} \mathtt{ope[-1,S[1,a],S[1,z]]} &== \mathtt{CO2[\{1/2\},\{1/2\}]*b[z]/(a-z)\hat{}(3/4)}, \\ \mathtt{ope[-1,S[1,a],S[-1,z]]} &== \mathtt{(a-z)\hat{}(-5/4) + res[0,b[a],c[z]]/(2*(a-z)\hat{}(1/4))}, \\ \mathtt{ope[-1,S[-1,a],S[-1,z]]} &== \mathtt{CO2[\{-1/2\},\{-1/2\}]*c[z]/(a-z)\hat{}(3/4)}, \\ \mathtt{ope[b[a],S[-1,z]]} &== \mathtt{CO2[\{1\},\{-1/2\}]*S[1,z]/(a-z)\hat{}(1/2)}, \\ \mathtt{ope[c[a],S[1,z]]} &== \mathtt{CO2[\{-1\},\{1/2\}]*S[-1,z]/(a-z)\hat{}(1/2)}. \end{aligned}$$

From the above, we should set

$$\texttt{CO2}[\{1/2\},\{1/2\}] = 1; \texttt{CO2}[\{-1/2\},\{-1/2\}] = 1;$$
$$\texttt{CO2}[\{1\},\{-1/2\}] = 1; \texttt{CO2}[\{1/2\},\{-1\}] = 1;$$

in order to realize $\oint_z \frac{daa^{-1/2}}{2\pi i} c(a) S_+(z) = S_-(z)$, etc.

5. Bosonization

bosonization	`bosonize[`*opr*`,z]`
bosonization rule	`BRule[`*f*`]`

We can use the function `bosonize[]` in order to bosonize the free fields, complex fermions and β, γ. `bosonize[`*opr*`,z]` gives a bosonized form of the operator *opr* according to the bosonization rule `BRule[`*f*`]`, by which we must set the bosonization rule for every elementary field f as

$$\texttt{BRule}[f[\texttt{z\_}]] := opr'[\texttt{z}];$$

Correct bosonization rules will lead us to the correct bosonization of composite operators.

5.1. Bosonization of a complex fermion

As a simple example, we present the bosonization of (b, c) fermions. To represent $b(z) = \exp(i\phi(z))$ and $c(z) = \exp(-i\phi(z))$, we set a bosonization rule

$$\texttt{BRule}[\texttt{bi}[\texttt{i\_}, 1, \texttt{z\_}]] := \texttt{vp}[\texttt{i}, \{1\}, \texttt{z}]; \texttt{BRule}[\texttt{ci}[\texttt{i\_}, 1, \texttt{z\_}]] := \texttt{vp}[\texttt{i}, \{-1\}, \texttt{z}];$$

then we have the bosonized form of the currents `Tbc[h,1,z]` and `Jc[1,z]` as

$$\texttt{bosonize}[\texttt{Tbc}[\texttt{h}, 1, \texttt{z}], \texttt{z}] == \texttt{Tpj}[2(2\texttt{h} - 1), 1, \texttt{z}], \ \texttt{bosonize}[\texttt{Jc}[1, \texttt{z}], \texttt{z}] == \texttt{Jp}[1, \texttt{z}].$$

5.2. Bosonization of a complex boson

Like a super-ghost of superstring, a complex boson (β,γ) can be bosonized. To represent $\beta(z)=i\partial\phi_1\exp(-i\phi_1(z)-\phi_2(z))$ and $\gamma(z)=\exp(i\phi(z)+\phi_2(z))$, we set a bosonization rule

```
BRule[be[i_, 1, z_]] := der[i, Nl[pj[0, 1, z], vp[0, {-1, I}, z]], z];
BRule[ga[i_, 1, z_]] := vp[i, {1, -I}, z];
```

then we have the bosonized form of the currents `Tbc[h,1,z]` and `Jc[1,z]` as

```
bosonize[Tbg[h, 1, z], z] == Tpj[1, 1, z] + Tpj[I(1 - 2h), 2, z],
bosonize[Jg[1, z], z] == -I Jp[2, z]
```

6. Examples

We will give examples for programing by discussing two-dimensional gravity coupled to minimal matter.

6.1. Free Field Realization of $osp(2|2)$ Lie Superalgebra

Bosonic generators J, H, E, F, and fermionic generators η^e_1, η^e_2, η^f_1, η^f_2 correspond to the following matrices;

$$
\begin{array}{llll}
J = -E_{12}+E_{21}, & H=E_{33}-E_{44}, & E=E_{34}, & F=E_{43}, \\
\eta^e_1 = E_{14}+E_{31}, & \eta^e_2 = E_{24}+E_{32}, & \eta^f_1 = -E_{13}+E_{41}, & \eta^f_2 = -E_{23}+E_{42}.
\end{array}
$$

The root diagram for $osp(2|2)$ is

$$
\begin{array}{ccccccccc}
 & & \Psi^f_+ & & & & \Psi^e_+ & & \\
 & \diagup & & \nwarrow & & \diagup & & \diagdown & \\
F & & — & & H,J & & \longrightarrow & & E \\
 & \diagdown & & \diagup & & \diagdown & & \diagup & \\
 & & \Psi^f_- & & & & \Psi^e_- & &
\end{array}
\quad , \qquad (6.1)
$$

where $\Psi^e_\pm$ and $\Psi^f_\pm$ correspond to the complex generators $\frac{\eta^e_1\pm i\eta^e_2}{\sqrt{2}}$ and $\frac{\eta^f_1\pm i\eta^f_2}{\sqrt{2}}$, respectively. For a choice of simple roots $\{\Psi^f_+, E\}$ which is indicated by the arrows in the diagram (6.1), we introduce free fields, (β,γ) for E, (b^f_+, c^f_-) for Ψ^f_+, (b^e_+, c^e_+) for Ψ^e_+, ϕ_H for H, and ϕ_J for J.

First, we set currents for positive roots and zero roots. Secondly, we find the current for the lowest root. Then we have all currents. We define currents for positive roots

$$\texttt{Psifplus}[\texttt{z\_}] := \texttt{bi}[0,1,\texttt{z}] + \texttt{Nl}[\texttt{bi}[0,2,\texttt{z}],\texttt{ga}[0,1,\texttt{z}]];$$
$$\texttt{Ee}[\texttt{z\_}] := \texttt{be}[0,1,\texttt{z}];$$
$$\texttt{Psieplus}[\texttt{z\_}] := \texttt{Module}[\{\texttt{a}\},\ \texttt{res}[-1,\texttt{Ee}[\texttt{a}],\texttt{Psifplus}[\texttt{z}]]];$$

which imply $\Psi^f_+ = b^f_+ + b^e_+\gamma$, $E = \beta$ and $\Psi^e_+ = b^e_+$. For zero roots

$$\texttt{J0}[\texttt{z\_}] := \texttt{Jc}[1,\texttt{z}] + \texttt{Jc}[2,\texttt{z}] - \texttt{I}\,\texttt{Q}\,\texttt{Jp}[2,\texttt{z}];$$
$$\texttt{H}[\texttt{z\_}] := 2\texttt{Jg}[1,\texttt{z}] - \texttt{Jc}[1,\texttt{z}] + \texttt{Jc}[2,\texttt{z}] + \texttt{Q}\,\texttt{Jp}[1,\texttt{z}];$$

which imply

$$J = b^f_+ c^f_- + b^e_+ c^e_- + \mathtt{Q}\partial\phi_J, \qquad H = -2\beta\gamma - b^f_+ c^f_- + b^e_+ c^e_- + i\mathtt{Q}\partial\phi_H.$$

Searching for the current of the lowest root is carried out as follows. Let $\texttt{Jformal}[\texttt{z\_}] := \texttt{Jc}[1,\texttt{z}] + \texttt{Jc}[2,\texttt{z}] + \texttt{Jg}[1,\texttt{z}] + \texttt{Jp}[1,\texttt{z}] + \texttt{Jp}[2,\texttt{z}];$ be a formal $u(1)$ current, and define the polynomial $\texttt{Lowest}[\texttt{z\_}] := \texttt{ci}[0,2,\texttt{z}] + \texttt{Nl}[\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}]]$, which have the same charges as Ψ^f_- and conformal weight 0. Using the function `Current[]`, we set the lowest root current as

$$\texttt{Psifminus}[\texttt{z\_}] := \texttt{Current}[\texttt{der}[\texttt{Lowest}[\texttt{z}],\texttt{z}] + \texttt{Nl}[\texttt{J}[\texttt{z}],\texttt{Lowest}[\texttt{z}]],\texttt{z},\texttt{f}];$$

It gives the following form

$$\begin{aligned}
&\texttt{f}[1]\,\texttt{ci}[1,2,\texttt{z}] + \texttt{f}[2]\,\texttt{Nl}[\texttt{ci}[0,2,\texttt{z}],\texttt{pj}[0,1,\texttt{z}]] + \texttt{f}[3]\,\texttt{Nl}[\texttt{ci}[0,2,\texttt{z}],\texttt{pj}[0,2,\texttt{z}]]+\\
&\texttt{f}[4]\,\texttt{Nl}[\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[1,1,\texttt{z}]] + \texttt{f}[5]\,\texttt{Nl}[\texttt{ga}[1,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}]]+\\
&\texttt{f}[6]\,\texttt{Nl}[\texttt{be}[0,1,\texttt{z}],\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,2,\texttt{z}]] + \texttt{f}[7]\,\texttt{Nl}[\texttt{bi}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}],\texttt{ci}[0,2,\texttt{z}]]+\\
&\texttt{f}[8]\,\texttt{Nl}[\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}],\texttt{pj}[0,1,\texttt{z}]] + \texttt{f}[9]\,\texttt{Nl}[\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}],\texttt{pj}[0,2,\texttt{z}]]+\\
&\texttt{f}[10]\,\texttt{Nl}[\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}],\texttt{bi}[0,2,\texttt{z}],\texttt{ci}[0,2,\texttt{z}]]+\\
&\texttt{f}[11]\,\texttt{Nl}[\texttt{be}[0,1,\texttt{z}],\texttt{ga}[0,1,\texttt{z}],\texttt{ga}[0,1,\texttt{z}],\texttt{ci}[0,1,\texttt{z}]];
\end{aligned}$$

Requiring that these satisfy the $osp(2|2)$ algebra, one can find a solution

$$\texttt{f}[1] = (-2+\texttt{Q\^{}2})/2; \texttt{f}[2] = \texttt{Q}/2; \texttt{f}[3] = \texttt{I}\,\texttt{Q}/2; \texttt{f4}[2] = -\texttt{Q\^{}2}/2; \texttt{f}[5] = 0;$$
$$\texttt{f}[6] = -1; \texttt{f}[7] = -1; \texttt{f}[8] = \texttt{Q}/2; \texttt{f}[9] = -\texttt{I}\,\texttt{Q}/2; \texttt{f}[10] = 0; \texttt{f}[11] = 0;$$

Other currents for the negative roots F and Ψ^e_- are given by

$$\texttt{Ff}[\texttt{z\_}] := \texttt{Module}[\{\texttt{a}\},\ \texttt{res}[-1,\texttt{Psifplus}[\texttt{a}],\texttt{Psifminus}[\texttt{z}]]];$$
$$\texttt{Psieminus}[\texttt{z\_}] := \texttt{Module}[\{\texttt{a}\},\ \texttt{res}[-1,\texttt{Ee}[\texttt{a}],\texttt{Psifminus}[\texttt{z}]]];$$

Corresponding to two simple roots, there are two screening currents S_E and S^f_+ which are Felder's BRST currents, of which zero modes commute with all generators of $osp(2|2)$.

They are given as

$$\texttt{SE}[\texttt{z\_}] := [\texttt{bi}[0,1,\texttt{z}], \texttt{vp}[0, \{1/\texttt{Q}, -\texttt{I}/\texttt{Q}\}, \texttt{z}]];$$
$$\texttt{Sf}[\texttt{z\_}] := \texttt{Nl}[\texttt{be}[0,1,\texttt{z}] - \texttt{Nl}[\texttt{bi}[0,2,\texttt{z}], \texttt{ci}[0,1,\texttt{z}]], \texttt{vp}[0, \{-2/\texttt{Q}, 0\}, \texttt{z}]];$$

6.2. Hamiltonian Reduction

Let us demonstrate how the Hamiltonian reduction method gives two-dimensional topological string from a free field realization of the Lie superalgebra $osp(2|2)$.

The Hamiltonian reduction to give the topological string is to impose the constraints $\Phi \approx 0$ for $\Phi \in \{E-1, \Psi^e_+, \Psi^e_-, H-J\}$ on the currents. From the primary constraints, we draw up the secondary constraints, which implies $0 = \langle \Phi |^{\forall} X \rangle$ as the operator relations. $\langle \Phi |^{\forall} X \rangle$ is given by the regular terms in the OPE $\Phi(a)X(z)$, i.e. `res[0,`$\Phi(a), X(z)$`]`.

The first two constraints $\{E-1, \Psi^e_+\}$ can always be imposed by the rule `redRule`

```
redRule = {Nl[b___, be[0, 1, z], c___] :> Fsort[Nl[b, c], z],
           Nl[b___, be[1, 1, z], c___] :> 0, Nl[b___, bi[i_, 2, z], c___] :> 0};
```

because these are linear in free fields. The last two constraints are used to eliminate terms depending on the fields γ (=`ga[0,1,z]`) and c^e_- (=`ci[0,2,z]`) and their derivatives.

```
redJ[z_] := J0[z]/.redRule;
redGplus[z_] := Psifplus[z]/.redRule;
redGminusBS[z_] := Fsort[Module[{a}, Psifminus[z]-
   res[0, H[a] - J0[a], Nl[ci[0, 2, z]]]/2 - (1 - Q^2/2)der[Psieminus[z], z]+
   Q/4 res[0, H[a] - J0[a], Nl[ci[0, 1, z], pj[0, 1, z]]]-
   I Q/4 res[0, H[a] - J0[a], Nl[ci[0, 1, z], pj[0, 2, z]]]-
   Q^2/4 res[0, H[a] - J0[a], Nl[ci[1, 1, z]]]]/.redRule, z];
redTBS[z_] := Fsort[Module[{a}, Ff[z] - res[0, H[a] - J0[a], Nl[ga[0, 1, z]]]/2-
  res[0, Psieminus[a], Nl[bi[0, 1, z]]] - I Q res[0, H[a] - J0[a], Nl[pj[0, 2, z]]]/4+
  Q res[0, H[a] - J0[a], Nl[pj[0, 1, z]]]/4 + Q^2der[H[z] - J0[z], z]/4]/.redRule, z];
```

These four currents correspond to the topological version of the ghost number current, the anti-ghost, the BRST-charge and stress-tensor current of the two dimensional bosonic string. They satisfy the twisted $N=2$ superconformal algebra.

Actually, 2 `redTBS[z]/Q^2` is equal to the total stress tensor,

$$\mathtt{T[z\_] := Tpj[(2 - Q\hat{}2)/Q, 1, z] + Tpj[(-I * (2 + Q\hat{}2))/Q, 2, z] + Tbc[2, 1, z];}$$

and the difference between `redGminusBS[z]` and the BRST charge which is defined by

$$\mathtt{BRST[z\_] := Nl[T[z] - Tbc[2, 1, z]/2, Nl[ci[0, 1, z]]];}$$

is just a total derivative;

$$\begin{aligned}&\mathtt{Fsort[redGminusBS[z] - Q\hat{}2/2BRST[z], z] == ci[2, 1, z] * (Q\hat{}2 - Q\hat{}4)/4-}\\ &\quad \mathtt{I/2 * der[Nl[ci[0, 1, z], pj[0, 2, z]], z] * Q\hat{}3;}\end{aligned}$$

Thus the zero-modes of the currents are the same. The R.H.S. of this equation is $\mathtt{Q}^2\partial((1-\mathtt{Q}^2)\partial c + 2\mathtt{Q}c\partial\phi_J)/4$. ϕ_J is thought to be a Liouville gravity field, because it appears in the topological version of the ghost number current `redJ`. The fields ϕ_H and ϕ_J are, then, matter degrees of freedom and Liouville degrees of freedom with central charges

$$c_M = 1 - 3\left(\frac{\mathtt{Q}^2 - 2}{\mathtt{Q}}\right)^2, \quad c_L = 1 + 3\left(\frac{\mathtt{Q}^2 + 2}{\mathtt{Q}}\right)^2.$$

Two screening currents S_E and S_+^f also remain screening currents through the Hamiltonian reduction. They are slightly changed to

$$\begin{aligned}&\mathtt{redSE[z\_] := Nl[bi[0, 1, z], vp[0, \{1/Q, -I/Q\}, z]];}\\ &\mathtt{redSf[z\_] := vp[0, \{-2/Q, 0\}, z];}\end{aligned}$$

There is another screening current that is one of the screening currents for the Feigin-Fuchs realization of matter,

$$\mathtt{Sf2[z\_] := vp[0, \{Q, 0\}, z];}$$

We can obtain the two-dimensional $N = 1$ superstring in the same manner starting from $osp(3|2)$, which is related to the free field realization of the $N = 3$ superconformal algebra discussed in ref. [9,10].

6.3. Two-dimensional Gravity coupled to minimal matter

Now, we consider two-dimensional gravity coupled to (p,q)-minimal matter. This is a short review of Kanno and Sarmadi [7], and for further details the reader is referred to Lian and Zuckerman [11] and Bershadsky *et al.* [10].

In the BRST-quantization procedure the physical states are defined to be the BRST cohomology class. For any CFT with the total central charge $c^{tot}=26$, we can introduce the (Virasoro) BRST complex with coefficients in the Virasoro module $\mathcal{M}^{c=26}$. The BRST operator

$$Q_B = \oint \frac{dz}{2\pi i} : \left(T + \frac{1}{2}T^G\right) c :$$

acts on the cochain space $C^*(Vir,\mathcal{M}) = \mathcal{M}\otimes\Lambda^{bc}$, where T is the total energy-momentum tensor for $\mathcal{M}$, and Λ^{bc} is the Fock module of the reparametrization ghosts (b,c) with the energy-momentum tensor T^G. The $\mathbf{Z}$-gradation of $C^*(Vir,\mathcal{M})$ is called ghost number. Let us consider the case $\mathcal{M} = L(c_{q,p},\Delta_{n',n})\otimes\mathcal{F}(Q^L,p^L)$ corresponding to 2D gravity coupled to the (q,p) minimal model. $L(c_{q,p},\Delta_{n',n})$ is the Virasoro irreducible module with central charge

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \qquad q<p, \qquad \text{coprime}$$

and the highest weight

$$\Delta_{m',m} = \frac{(qm'-pm)^2-(p-q)^2}{4pq},$$

where $1\le m'\le p-1$ and $1\le m\le q-1$. Comparing this central charge with the matter central charge c_M in the previous section, the value of Q can be given by $-\sqrt{\frac{2q}{p}}$. We assume a free field realization of the Liouville field, and the Fock module of the Liouville field is given by the Feigin-Fuchs module with the background charge Q_L and vacuum momentum p_L. The central charge and the Virasoro highest weight are given by

$$c_L = 1+12Q_L^2, \quad h(p_L) = -\frac{1}{2}p_L^2 + p_LQ_L.$$

Comparing this with the expression in the previous section, we can set $Q_L = -\frac{\mathsf{Q}}{2} - \frac{1}{\mathsf{Q}} = \frac{\beta_0}{2}$, where we introduced $\beta_+ = \sqrt{\frac{2p}{q}}$, $\beta_- = \sqrt{\frac{2q}{p}}$ and $\beta_0 = \beta_+ + \beta_-$. Let us introduce the standard notation of the Liouville momentum $\beta_{n',n} = \frac{1-n}{2}\beta_+ + \frac{1-n'}{2}\beta_-$.

Lian and Zuckerman found that there are infinitely many physical (that means BRST-invariant) states with non-trivial ghost number for every primary $\Delta_{m',m}$. These states are given as follows. Let $E_{m',m}(q,p) = \{a_t, b_t\}_{t\in\mathbf{Z}}$ denote the set of highest weights appearing in the embedding diagram of Verma modules;

$$\begin{array}{ccccccc} M(c,b_0) & \leftarrow & M(c,a_{-1}) & \leftarrow & M(c,b_{-1}) & \leftarrow & M(c,a_{-2})\dots \\ & \nwarrow & & \nwarrow\!\!\!\!\!\swarrow\!\!\!\!\!\nearrow\!\!\!\!\!\searrow & & \nwarrow\!\!\!\!\!\swarrow\!\!\!\!\!\nearrow\!\!\!\!\!\searrow & \\ & & M(c,a_0) & \leftarrow & M(c,b_1) & \leftarrow & M(c,a_1)\dots, \end{array}$$

where

$$a_t = \frac{(2qpt + qm' + pm)^2 - (q-p)^2}{4qp}, \quad b_t = \frac{(2qpt + qm' - pm)^2 - (q-p)^2}{4qp}.$$

The relative cohomology $H^*_{rel}(Vir, \mathcal{M})$ is non-vanishing if and only if the Liouville momentum p_L satisfies

$$1 - h(p_L) \in E_{m',m}(q,p). \tag{6.2}$$

$H^*_{rel}(Vir, \mathcal{M})$ is at most one-dimensional. That is, we have a unique physical state (Lian-Zuckerman state) for each p_L satisfying $1 - h(p_L) \in E_{m',m}(q,p)$. The ghost number is given by

$$n_{gh} = \pi(p_L) d(p_L),$$

where $d(p_L)$ is the number of arrows leading from $b_0 = \Delta_{m',m}$ to $1 - h(p_L)$ in the embedding diagram, and $\pi(p_L) = sign(p_L - Q_L)$.

Solving the Eq.(6.2) for every element of $E_{m',m}(q,p)$, we obtain the following 'spectrum' of the Liouville momentum:

$$\beta^{\pm}(a_t) = \frac{\beta_0 \pm |(tq+m)\beta_+ + (tp+m')\beta_-|}{2}, \quad \beta^{\pm}(b_t) = \frac{\beta_0 \pm |(tq-m)\beta_+ + (tp+m')\beta_-|}{2}.$$

Let k be a positive integer. The ghost number n_{gh} for each Liouville momentum is given by:

$$\begin{array}{rcccl} \beta^+(a_{k-1}), & \beta^+(a_{-k}) & \rightarrow & n_{gh} = & 2k, \\ \beta^+(b_k), & \beta^+(b_{-k}) & \rightarrow & n_{gh} = & 2k+1\ , \\ \beta^-(a_{k-1}), & \beta^-(a_{-k}) & \rightarrow & n_{gh} = & -2k+2, \\ \beta^-(b_k), & \beta^-(b_{-k}) & \rightarrow & n_{gh} = & -2k+1, \\ \beta^+(b_0), & \beta^-(b_0) & \rightarrow & n_{gh} = & 1. \end{array}$$

Using the condition $1 \le m \le q-1, 1 \le m' \le p-1$, every Liouville momentum of $\beta^{\pm}(a_t)$

and $\beta^{\pm}(b_t)$ can be rewritten as

$$\beta(s,s') = -n_{gh}\beta_w + s'\beta_x + s\beta_y, \quad (0 \le s \le q-2,\ 0 \le s' \le p-2), \tag{6.3}$$

where we have defined

$$\beta_w = \beta_{p+1,1} = \beta_{1,q+1}, \quad \beta_x = \beta_{2,1}, \quad \beta_y = \beta_{1,2}.$$

This gives the dimension of the observables with ghost number n_{gh} in 2D gravity coupled to the (q,p) model, that is $(q-1)(p-1)$.

This linear spectrum (6.3) means a possible identification

$$\mathcal{O}[\beta(s,s')] = w^{-n_{gh}} x^{s'} y^{s},$$

where w, x and y denote the observables with the momenta β_w, β_x and β_y, respectively. In the free field description of the matter sector, they are given by the following expressions:

$$x = (bc + (i\alpha_{1,2}\partial X + \beta_{1,2}\partial\phi))\, e^{i\alpha_{2,1}X + \beta_{2,1}\phi},$$
$$y = (bc + (i\alpha_{2,1}\partial X + \beta_{2,1}\partial\phi))\, e^{i\alpha_{1,2}X + \beta_{1,2}\phi},$$

where $\alpha_{n',n} = \frac{1-n}{2}\alpha_+ + \frac{1-n'}{2}\alpha_-$, $\alpha_+ = \beta_+$ and $\alpha_- = -\beta_-$. (The background matter charge is equal to $-\alpha_0$ where $\alpha_0 = \alpha_+ + \alpha_-$.)

For w, because w has ghost number $n_{gh} = -1$ and conformal weight 0, solving $\beta^-(b_1) = \beta_w$ gives $\alpha_w = \alpha_{1,q-1} = \frac{(2-q)\beta_+}{2}$, and thus we have

$$w = \mathcal{P} e^{i\alpha_{1,q-1}X + \beta_{1,q+1}\phi}, \quad \mathcal{P} = \partial^{p+q-3} b + \cdots,$$

where $\mathcal{P}$ is a differential polynomial with conformal weight $p+q-1$ and ghost number -1. We must find the polynomial $\mathcal{P}$ s.t. w is BRST invariant.

For w^{-1}, because w^{-1} has ghost number $n_{gh} = 1$ and conformal weight 0, solving $\beta^+(b_1) = -\beta_w$ gives $\alpha_{w^{-1}} = \alpha_{p-1,1} = \frac{(p-2)\beta_-}{2}$, and thus we have

$$w^{-1} = c e^{i\alpha_{p-1,1}X - \beta_{1,q+1}\phi}.$$

The action of w on w^{-1} gives $e^{i\alpha_0 X}$, which implies that w^{-1} is exactly the inverse of w. The existence of the ring generators $\{w, w^{-1}\}$ is the main subject of Kanno and Sarmadi [7]. The linear spectrum (6.3) strongly suggests that there is a ring structure of the observables, of which elements are represented as $w^{-n_{gh}}x^iy^j$ for $0 \le i \le p-1$, $0 \le j \le q-1$. And $x^{p-1} = y^{q-1} = 0$ is expected to be valid up to the BRST-exact terms, because they are outside the Kac table. To obtain the absolute cohomology from the relative one, we only introduce a BRST invariant operator

$$a = c\partial\phi + \frac{1}{2}\beta_0\partial c = [Q_B, \phi]. \tag{6.4}$$

Thus, the cohomology is made of elements $w^{-n_{gh}}x^iy^j$ and $aw^{-n_{gh}}x^iy^j$.

Now, we shall summarize the ring generators and the screening charges of our realization of the model. We have the generators of the ring, $\{x, y, w, w^{-1}, a\}$. The screening currents are following:

$$\begin{array}{rrcllcl} \text{Liouville:} & s_+^L & = & e^{\beta_+\phi}, & s_-^L & = & e^{\beta_-\phi}, \\ \text{matter:} & s_+^M & = & e^{i\beta_+X}, & s_-^M & = & e^{-i\beta_-X}, \\ \text{mixed:} & s_x & = & be^{i\frac{\beta_-}{2}X-\frac{\beta_-}{2}\phi}, & s_y & = & be^{-i\frac{\beta_+}{2}X-\frac{\beta_+}{2}\phi}. \end{array}$$

Among these, s_+^M, s_-^M and s_y are screening currents also in the topological version of the algebra, and s_+^M and s_y are screening originally in the Lie superalgebra version of the algebra. The Felder BRST resolution can be discussed by using these screening currents. The screening currents s_x and s_y are related to the rings x and y by

$$\left\{\int s_x, a\right\} = -\frac{\beta_-}{2}x, \quad \left\{\int s_y, a\right\} = -\frac{\beta_+}{2}y,$$

which does not mean that the rings x and y are Felder BRST trivial. By Eq. (6.4), x and y can be expressed in the 'BRST exact' form, which, however, does not mean that they are BRST trivial. The reason why we do not think that the elements of the ring are trivial is that they are BRST-variations of some unphysical operators.

To demonstrate the vanishing of x^{p-1} and y^{q-1}, we can always find special tachyonic states $\mathcal{V}_x$ and $\mathcal{V}_y$:

$$\begin{aligned} x^{p-1}\mathcal{V}_x &= 0, & \mathcal{V}_x &= ce^{i\alpha_{2,q}X+\beta_{p,2}\phi}, \\ y^{q-1}\mathcal{V}_y &= 0, & \mathcal{V}_y &= ce^{i\alpha_{p,2}X+\beta_{2,q}\phi}, \end{aligned}$$

where $\mathcal{V}_x$ and $\mathcal{V}_y$ are outside the Kac table. The implication of this is not yet explained.

We should remark how we compute elements of the ring. For w^{2n}, solving $p_L = 2n\beta_w$ and $n_{gh} = -2n$, we find $p_M = 0, \alpha_0$. We write $\widetilde{w^{2n}}$ for w^{2n} with $p_M = \alpha_0$. For w^{2n+1}, solving $p_L = (2n+1)\beta_w$ and $n_{gh} = -2n-1$, we find $p_M = \alpha_{1,q-1}, \alpha_{p-1,1}$. We write $\widetilde{w^{2n+1}}$ for w^{2n+1} with $p_M = \alpha_{p-1,1}$. We can understand the procedure to obtain w^{2n} and w^{2n+1} from the diagram

$$\begin{array}{ccccc} w^{2n} & \Rightarrow & w^{2n+1} & \Leftarrow & w^{2n+2} \\ \Uparrow & \searrow\!\!\!\swarrow & \uparrow & \searrow\!\!\!\swarrow & \Uparrow \\ \widetilde{w^{2n}} & \Leftarrow & \widetilde{w^{2n+1}} & \Rightarrow & \widetilde{w^{2n+2}} \end{array}.$$

The arrow "$\Uparrow$" is the action of a charge $Q_{p-1,q-1}$, where

$$Q_{n,n'} = \int \prod_{i,j} dv_i du_j \, s_-^M(v_n) \cdots s_-^M(v_1) s_+^M(u_{n'}) \cdots s_+^M(u_1),$$

which has the momentum $p^M = n'\beta_+ - n\beta_-$. The arrow "$\uparrow$" is the action of a charge $Q_{1,q-1}$. The arrow "$\searrow$" is the action of w. The arrow "$\Rightarrow$" is the action of the dual of w. The arrow "$\swarrow$" is the action of w^{-1}. The arrow "$\Leftarrow$" is the action of the dual of w^{-1}.

While the actions of x and y on w^{2n} with $p_M = 0$ behave well, the actions of x on $\widetilde{w^{2n+1}}$ with $p_M = \alpha_{p-1,1}$ and y on w^{2n+1} with $p_M = \alpha_{1,q-1}$ are not well defined. Thus, to compute the OPE of y and $x^i y^j w^{2n+1}$, we have to screen $x^i y^j w^{2n+1}$ to $Q_{1,q-1} x^i y^j w^{2n+1} (= x^i \widetilde{y^j w^{2n+1}}$, a dual operator of $x^i y^j w^{2n+1}$), and compute the action of y on it, and screen again $Q_{1,q-1} x^i y^j w^{2n+1}$ by a charge $Q_{p-1,1}$ to give $x^i y^{j+1} w^{2n+1}$.

The action of the $Q_{n,n'}$ cannot be computed only by considering the OPE of operators, because the definition of $Q_{n,n'}$ should contain the information of the non-trivial integral region.

6.4. Chiral Symmetry Algebra

If $\psi^{(n)}$ is a physical operator of ghost number n, then $j^{(n-1)}(z) = \oint dz' b(z') \psi^{(n)}(z)$ is an operator of ghost number $(n-1)$ and conformal weight 1. The BRST variation of this operator is a total derivative of some operator up to null operators. This implies that

$$\left\{ Q_B, \oint dz' j^{(n)}(z') \tilde{\psi}^{(m)} \right\} = \oint dz' \frac{\partial}{\partial z'} (*) = 0,$$

for a physical operator $\tilde{\psi}^{(m)}$ up to null operators. Then $\oint dz' j^{(n)}(z') \tilde{\psi}^{(m)}$ is thought to be a physical operator with ghost number (n+m), if it is not BRST-exact.

There is such a current for each physical operator, which is generated by the ring generators $\{a, x, y, w, w^{-1}\}$. The action of the currents on the physical operators are computable, and that can be represented by the vector fields. For $ax^i y^j w^n$ and $x^i y^j w^n$,

$$K_n^{i,j} = \sum_{v=w,x,y,a} (C_v)_n^{i,j} x^i y^j w^n v \partial_v, \qquad G_n^{i,j} = x^i y^j w^n \partial_a,$$

where $(C_v)_n^{i,j}$ is a c-number coefficient. Kanno and Sarmadi [7] have remarked that the coefficients should be $(C_w)_n^{i,j} = -1$, $(C_x)_n^{i,j} = -\frac{1}{p}$, $(C_y)_n^{i,j} = -\frac{1}{q}$ and $(C_a)_n^{i,j} = n + \frac{i}{p} + \frac{j}{q}$. One can verify that by the action of $K_0^{i,j} (i, j = 0, 1)$ on a, x, y and w^{-1}, for general p, q. Let `x[z]`, `y[z]`, `a[z]` and `Winv[z]` correspond to x, y, a and w^{-1}, respectively. Then, for `K000[z]` which is defined by `res[-1, bi[0, 1, b], a[z]]`, we have

res[-1,K000[b],a[z]]==0, res[-1,K000[b],x[z]]== $-\frac{1}{p}$x[z],

res[-1,K000[b],y[z]]== $-\frac{1}{q}$y[z], res[-1,K000[b],Winv[z]]== $-$Winv[z].

For `K100[z]` which is defined by `res[-1, bi[0, 1, b], res[0, x[c], a[z]]]`, we have

res[-1,K100[b],a[z]]== $\frac{1}{p}$res[0,x[c],a[z]],

res[-1,K100[b],x[z]]== $-\frac{1}{p}$res[0,x[c],x[z]],

res[-1,K100[b],y[z]]== $-\frac{1}{q}$res[0,x[c],y[z]],

res[-1,K100[b],Winv[z]]== $-$res[0,x[c],Winv[z]].

For `K010[z]` which is defined by `res[-1, bi[0, 1, b], res[0, y[c], a[z]]]`, we have

res[-1,K010[b],a[z]]== $\frac{1}{p}$res[0,y[c],a[z]],

res[-1,K010[b],x[z]]== $-\frac{1}{p}$res[0,y[c],x[z]],

res[-1,K010[b],y[z]]== $-\frac{1}{q}$res[0,y[c],y[z]],

res[-1,K010[b],Winv[z]]== $-$res[0,y[c],Winv[z]].

For `K00m1[z]` which is defined by `res[-1, bi[0, 1, b], res[0, Winv[c], a[z]]]`, we have

res[-1,K00m1[b],a[z]]== - res[0,Winv[c],a[z]],

res[-1,K00m1[b],y[z]]== $-\frac{1}{q}$res[0,Winv[c],y[z]].

Some of the equations in the above are correct only up to BRST-exact terms.

Corresponding to w, we have a current $j^{(-2)} = \oint dz' b(z') w$, which has the same matter and Liouville momentum as those of w, and its zero mode commutes with the BRST charge Q_B. Therefore, the current is of the form

$$j^{(-2)} = \mathcal{P} e^{i\alpha_{1,q-1} X + \beta_{1,q+1}\phi}, \quad \mathcal{P} = \partial^{p+q-4} bb + \cdots,$$

which is easier to construct than w itself because the power of the derivative, $p + q - 4$, is lower than $p + q - 3$ for that of w. The action of $\oint dz j^{(-2)}$ on the state aw^{-1} should give $e^{i\alpha_0 X}$. The action on a will give w.

<u>(2,3) model</u>. The operator w is given by $w = \mathcal{P}e^{i\alpha_{1,q-1}X+\beta_{1,q+1}\phi}$, where $\mathcal{P} = \partial^{p+q-3}b+\cdots$. The BRST-invariance of w ($\{Q_B, w\} = 0$ up to null vectors) requires that the polynomial is given as $\mathcal{P} = \mathcal{L}_0 + \mathcal{L}_1$, where

$$\begin{aligned} \mathcal{L}_0 =& \quad \partial^2 b - \sqrt{3}(2b\partial^2\phi - \partial b\partial\phi) - 6\partial bbc, \\ \mathcal{L}_1 =& \quad a^w b\left(-\partial X\partial X + \frac{i\partial^2 X}{\sqrt{3}}\right), \end{aligned}$$

where $\mathcal{L}_1 e^{i\alpha_{1,q-1}X+\beta_{1,q+1}\phi}$ is a null vector and a^w is a constant undetermined, and we can set it zero. The action w on w^{-1} gives $ww^{-1} = e^{i\alpha_0 X}$. Then, w^{-1} is actually the inverse of w.

The action of the current $j^{(-2)} = -2\sqrt{3}b\partial be^{i\alpha_{1,q-1}X+\beta_{1,q+1}\phi}$ on a gives $\mathcal{L}_0 e^{i\alpha_{1,q-1}X+\beta_{1,q+1}\phi}$ only.

<u>(3,4) model</u>. If we try to construct w directly, the length of the polynomial $\mathcal{P}$ is so long that we cannot compute it in a short time. Instead, we construct $j^{(-2)} = \mathcal{J}e^{i\alpha_{1,q-1}X+\beta_{1,q+1}\phi}$, where

$$\begin{aligned} \mathcal{J} =& \quad \partial^3 bb - \tfrac{2}{3}\partial^2 b\partial b + \tfrac{2\sqrt{6}i}{9}\partial bb\partial^2 X - \tfrac{4\sqrt{6}}{3}\partial bb\partial^2\phi - \tfrac{2\sqrt{6}i}{3}\partial^2 bb\partial X \\ & -\tfrac{4\sqrt{6}}{3}\partial^2 bb\partial\phi + \tfrac{4}{9}\partial bb\partial X\partial X + \tfrac{8i}{3}\partial bb\partial X\partial\phi + 4\partial bb\partial\phi\partial\phi \end{aligned}$$

The action of $\oint j^{(-2)}$ on aw^{-1} gives $\frac{5\sqrt{6}}{18}e^{i\alpha_0 X}$. Then, w^{-1} is actually the inverse of w. The length of the w is 46 terms long, if we compute $\oint dz' j^{(-2)}(z')a(z)$.

7. Remarks and Plans

We remark on the limitation of the use of this package, "ope.math". This package is programed as a package for the *Mathematica 2.1 for HP 9000 RISC*, and has been checked on the *HP-UX A.09.01 E 9000/730* of the theory group of *KEK*.

We plan some applications and further developments. Because of the complication in computing OPEs of the W-algebras, we need this kind of tool to calculate OPEs. We will compute such complicated algebras.

So far, we could not make use of this package for computing Felder BRST-cohomology directly because of the difficulty of defining the integral contour to give the Felder BRST charge. We will try to construct an algorithm for this.

To compute correlation functions, we need to compute an OPE not in an expansion form but in a full form. We should develop the package to do that.

We expect that the package will help to study string theory, conformal field theory, topological field theory, two-dimensional gravity and other subjects, which are related to conformal field theories.

If you are interested in "ope.math", you can get it by e-mail from the author. Questions, remarks and improvements are welcome.

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