A GRADUATE TEXT FOR THE CORE COURSE
– GRAPH THEORY –

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Abstract. The purpose of the text is to give a brief overview of graph theory, from a mathematical viewpoint. Starting with basic concepts, we proceed to various important topics, which are carefully selected to enjoy the colorful world of graph theory. We study many familiar theorems and several uncommon theorems originally found for this text. We supply possibly many figures and examples to help readers understand the contents.

1 Graphs 1.1 definition

A pair $G = (V, E)$ of sets is called a graph or a graph on V if E consists of some sets of distinct two elements of V. Then V is called the vertex set (the set of all vertices) of G, and E is called the edge set (the set of all edges) of G. The components V and E of G are often denoted by $V(G)$ and $E(G)$, respectively. An edge $e = \{u, v\}$ (uv for short) is said to connect or join u and v, or u and v are called the end vertices of e. We usually represent a graph by a figure, a vertex by a dot or a small circle, and an edge uv by a line or a curve between u and v, where edges can cross each other. An edgeless graph or an independent set is a graph with no edges. In particular, a graph with no vertices and edges is called the empty graph, denoted by $(\varnothing, \varnothing)$ or simply $\varnothing$, but for simplicity, we usually exclude it from our thought. If V is finite then G is called finite, otherwise G is called infinite.

It is sometimes convenient to allow a "graph" to have multiple edges (multi-edges), loops or multiple loops (multi-loops). A "graph" which can have multi-edges and (multi-)loops is called a multigraph. Here, multi-edges are all edges having the same pair of end vertices, a loop is an edge that connects a vertex u to itself, written as uu, and multi-loops are all loops having the same end vertex. A multi-edge (respectively, multi-loop) refers to each single edge (respectively, loop) contained in some multi-edges (respectively, multi-loops). Note that a loop is a kind of an edge in a multigraph. In this text, a graph is assumed to have no multi-edges or loops, and so every graph is a multigraph but not conversely.

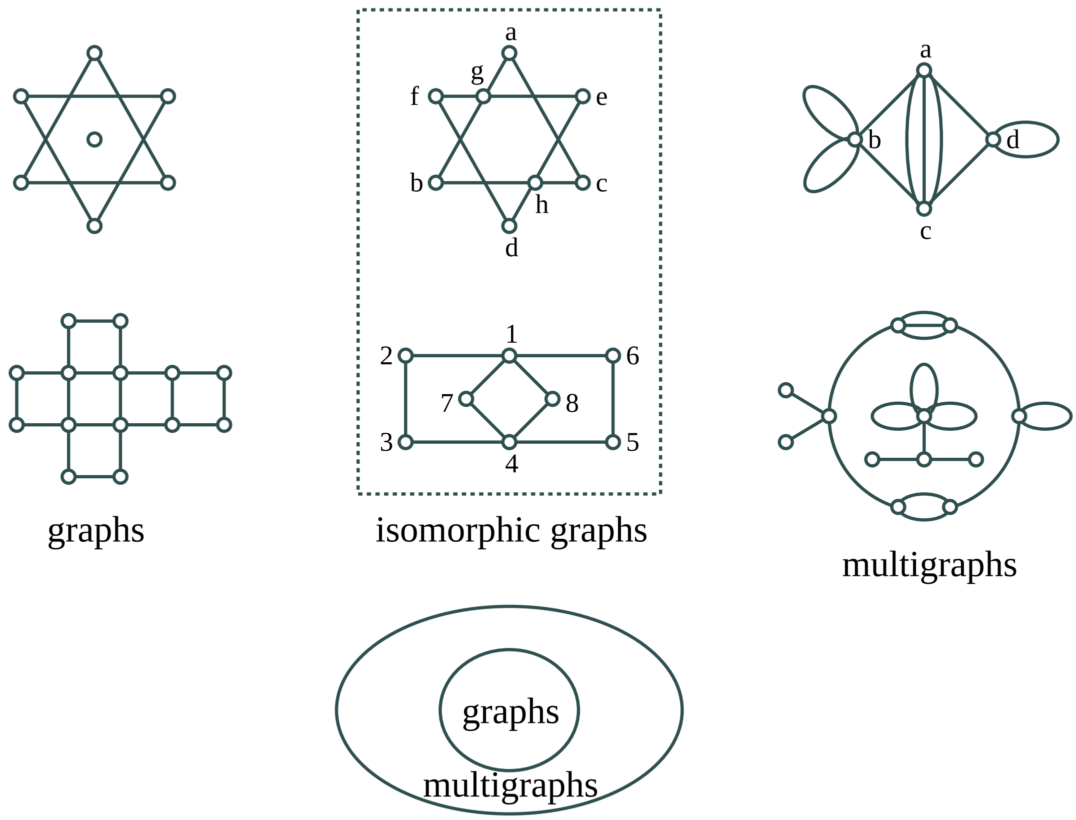


Figure 1. Graphs and multigraphs

Arbitrary multi-edges or multi-loops are regarded as distinct from each other, thus to be exact, we give a unique name to every multi-edge or multi-loop as $e : uv$ or $f : uu$, respectively. For simplicity, however, we sometimes omit their names if confusion does not occur. For a multigraph $G = (V, E)$, the edge set E is really a set when edge names are given, whereas E is a multiset when edge names are omitted.

The multiplicity of an edge uv (possibly $u = v$) is the number of edges uv, and the multiplicity of a multigraph is the maximum multiplicity of its edges. The order of a multigraph G is the cardinality of its vertices, and the size of G is the cardinality of its edges. A multigraph of finite order and size is called finite, while one of infinite order and/or size is called infinite. We assume that all multigraphs are finite, unless specifically stated otherwise.

For a (multi)graph, vertices u and v (possibly $u = v$) are called adjacent, denoted by $u \sim v$, if an edge $e = uv$ exists. Then e is called incident to the vertices u and v, or u and v are incident to e. If an edge $e = uv$ does not exist, then u and v are called independent or u is independent from v. A vertex adjacent to v is called a neighbor of v. Two distinct edges with at least one common end vertex are called adjacent. A set of vertices or of edges is called independent if no two of its elements are adjacent.

1.2 subgraphs

Let $G = (V, E)$ be a (multi)graph. A (multi)graph $G' = (V', E')$ made by taking subsets $V' \subset V$ and $E' \subset E$ is called a sub(multi)graph of G, where E' contains only some edges whose end vertices belong to V'. Then G is called a super(multi)graph of G'. By definition, G itself is a sub(multi)graph of G, and the other sub(multi)graphs are called proper. In this section, hereafter definitions will be given only for graphs, but can be given for multigraphs similarly.

A subgraph H of G is called a spanning subgraph or a factor of G if $V(H) = V$. We say H spans G. Let S be a subset of V. A subgraph H of G such that $V(H) = S$ with the maximum edge set is uniquely determined by S, and is said to be an induced subgraph of G, or induced by S, denoted by $G[S]$.

We define a (partial) order over the set of all subgraphs of G by

$$H \le H' \iff H \text{ is a subgraph of } H'. \tag{1}$$

Then $G[S]$ is nothing but the maximum subgraph H satisfying $V(H) = S$.

For a subset V' of V, we denote by $G - V'$ a graph made by removing all vertices in V' from G. Similarly, for a subset E' of E, $G - E'$ is a graph made by removing all edges in E' from G. For a vertex or an edge x of G, we sometimes write $G - \{x\} = G - x$ for short. Further, for a subgraph H of G, $G - H$ is a graph made by removing H from G, it is nothing but $G - V(H)$.

The subgraph of G induced by the set S of all neighbors of v (not including v itself) is called the (open) neighborhood of v and denoted by $N_G(v)$. If S contains v itself, then the subgraph induced by S is called the closed neighborhood of v, denoted by $N_G[v]$. When stated without any qualification, a neighborhood is assumed to be open. The subscript G is usually dropped when there is no danger of confusion.

1.3 isomorphic graphs

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A mapping $\phi : V \longrightarrow V'$ is called an isomorphism from G to G' if ϕ is a bijection and satisfies

$$u \text{ and } v \text{ are adjacent in } G \iff \phi(u) \text{ and } \phi(v) \text{ are adjacent in } G'. \tag{2}$$

For multigraphs G and G', an isomorphism from G to G' is defined to be a bijection $\phi : V \longrightarrow V'$ such that, for every $u, v \in V$,

$$\begin{array}{l} \text{the number of edges } uv = \text{the number of edges } \phi(u)\phi(v) \\ \text{the number of loops } uu = \text{the number of loops } \phi(u)\phi(u). \end{array} \tag{3}$$

Two (multi)graphs G and G' are said to be isomorphic, denoted by $G \simeq G'$, if there exists an isomorphism from G to G'. We see as usual that:

$$\begin{array}{l} G \simeq G, \qquad G \simeq G' \iff G' \simeq G, \\ G \simeq G',\ G' \simeq G'' \implies G \simeq G''. \end{array} \tag{4}$$

An isomorphism from G to itself is called an automorphism of G. The set of all automorphisms of G forms a group with composition of automorphisms, which is called the automorphism group of G.

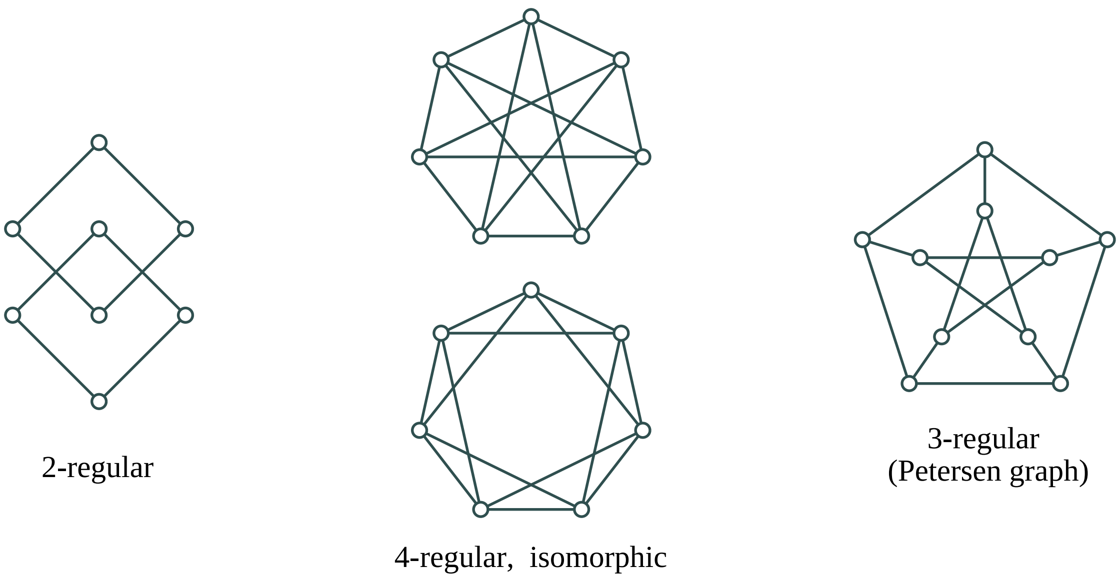


FIGURE 2. Examples of regular graphs

(exercise) Write the vertex sets and the edge sets of the (multi)graphs in Figure 1 (giving names of vertices, if needed).

(exercise) Find several (induced) sub(multi)graphs of the (multi)graphs in Figure 1.

(exercise) Find an isomorphism between the graphs in the center of Figure 1.

(ans) For example:

(5)

v	a	b	c	d	e	f	g	h
$\phi(v)$	6	7	5	3	8	2	1	4

(exercise) Determine the automorphism group of independent n vertices.

1.4 the degrees of vertices

Let G be a multigraph. The degree of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v, with loops vv being counted twice. We have the following.

Theorem 1.1. *(Handshaking lemma) Let G be a multigraph with the vertices $v_1, \ldots, v_p$ and q edges, then*

$$\sum_{i=1}^{p} \deg(v_i) = 2q. \tag{6}$$

Proof. Every non-loop edge is incident to distinct two vertices, and every loop is incident to one vertex. Thus if one counts up the degrees of all vertices, then every edge is counted twice. $\square$

A vertex of even (respectively, odd) degree is called an even (respectively, odd) vertex. A vertex of degree 0 is called isolated. A vertex of degree 1 is called a leaf or pendant (vertex). The maximum degree $\Delta(G)$ of G is the largest degree over all vertices; the minimum degree $\delta(G)$, the smallest.

A graph consisting of a single isolated vertex is called a trivial (singleton) graph. A multigraph in which every vertex has the same degree (k) is called (k-)regular. A 0-regular graph is an independent set. A 1-regular graph is a matching. A 2-regular multigraph is a vertex disjoint union of cycles. A 3-regular multigraph is said to be cubic, or trivalent. A k-factor is a k-regular spanning submultigraph. A 1-factor is a perfect matching. A 2-factor is spanning cycles. A partition of edges of a multigraph into k-factors is called a k-factorization. A k-factorable multigraph is one that admits a k-factorization.

The weakly decreasing sequence[1] of the degrees of all vertices of G is called the degree sequence of G. Isomorphic graphs have the same degree sequence, but the converse does not hold. A sequence of (usually weakly decreasing) integers is called a graphic(al) sequence if (the weakly decreasing arrangement of) it is the degree sequence of some graph.

Theorem 1.2. *A weakly decreasing sequence $d_1, d_2, \dots, d_n$ is graphic if and only if $d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ is graphic (allowed $d_{d_1+1} - 1 < d_{d_1+2}$).*

Proof. Let $d_1, d_2, \dots, d_n$ be a weakly decreasing sequence of nonnegative integers. Suppose $d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ is graphic. Let H be a graph whose degree sequence is the weakly decreasing arrangement of $d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$, that correspond to the vertices $v_2, \dots, v_n$, respectively. We can construct a graph G by adding a new vertex v_1 and edges $v_1 v_k$ ($k = 2, \dots, d_1 + 1$), so that G has the degree sequence $d_1, \dots, d_n$. Hence this sequence is graphic as desired.

Conversely, let G be a graph with the degree sequence $d_1, \dots, d_n$ corresponding again to $v_1, \dots, v_n$. If the first d_1 large degree vertices $v_2, v_3, \dots, v_{d_1+1}$ except v_1 are adjacent to v_1, then we can remove v_1 and the edges incident to v_1 to get a graph H whose degree sequence is the weakly decreasing arrangement of $d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$. Otherwise, we have the first vertex v_i independent from v_1, and after v_i, we have the first vertex v_j adjacent to v_1, where $d_i \geq d_j$. Then there exists a vertex v_k adjacent to v_i and independent from v_j. (If not, we have $d_i < d_j$, contradiction.) Now remove the edges $v_1 v_j$, $v_i v_k$, and add the edges $v_1 v_i$, $v_j v_k$. This process does not change the degree sequence, but it pushes back the position of the first vertex independent from v_1. Repeating this process, we reach the first case. □

(exercise) Find two non-isomorphic graphs with the degree sequence 2,2,2,2,2,2.

(exercise) Find all non-isomorphic graphs with the degree sequence 2,2,2,2,2,2,2,2,2.

(exercise) Find several non-isomorphic graphs with the degree sequence 3,3,2,2,1,1.

(exercise) Which are graphic? (1) 8,7,7,6,5,3,2,2,2,2,1,1. (2) 6,6,6,6,6,6,6,6,6,6. (3) 9,8,7,7,7,6,5,4,2,1. (4) 5,5,5,4,4,4,3,3,3,2,2,2,1,1,1. (5) 9,9,9,8,7,6,5,5,4,3,2,1,1,1.

[1] A sequence $a_0, a_1, a_2 \dots$ is called weakly (respectively, strictly) decreasing if $a_i \geq a_{i+1}$ (respectively, $a_1 > a_{i+1}$) for all i, and it is called weakly (respectively, strictly) increasing if $a_i \leq a_{i+1}$ (respectively, $a_1 < a_{i+1}$) for all i.

(exercise) Prove $n, n, n-1, n-1, \ldots, 1, 1$ is graphic.

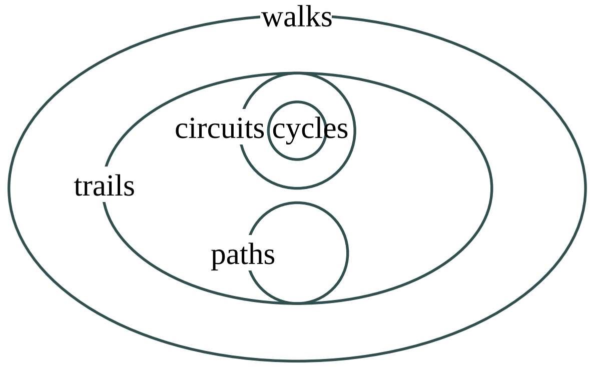


FIGURE 3

1.5 walks, trails, paths

Let G be a multigraph. For $n \geq 0$, an alternating sequence of vertices and edges of G, beginning and ending with vertices:

$$w = v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n, \tag{7}$$

where e_i connects v_{i-1} and v_i for every $i = 1, \ldots, n$, is called a walk of length n from v_0 to v_n (between v_0 and v_n) in G. Here v_0 and v_n are connected by w, and are called the initial and terminal vertices of w, respectively, or simply the end vertices (ends) of w. The vertices $v_1, v_2, \ldots, v_{n-1}$ are called inner vertices of w. We say w passes (through) each of $v_0, \ldots, v_n$ and $e_1, \ldots, e_n$. For a submultigraph H, if w passes some vertex or edge of H, then we say w meets H. For a set of vertices X or a set of edges F, if w passes some element of X (respectively, F), then we say w meets X (respectively, F). If $v_0 = v_n$, w is called closed, otherwise w is called open. In a walk, we can use vertices and edges repeatedly. Restricting walks by several conditions, we have the following classification.

a walk	no restrictions
a trail	every edge can be used at most once
a path	every vertex can be used at most once
a circuit	a closed trail of length ≥ 1
a cycle	a circuit with distinct vertices except for the end vertices

A cycle of length n is called an n-cycle, which is denoted by C_n. An even (respectively, odd) cycle is defined to be a cycle of even (respectively, odd) length. One theorem is that a graph is bipartite if and only if it contains no odd cycles. A path of length $n-1$ (on n vertices) is denoted by P_n. The symbols C_n and P_n are also used for subgraphs that they pass through.

A path is a trail, and a trail is a walk. A trivial path is a path of length 0, a path from a vertex to itself. A cycle is a circuit. The minimum length of cycles is 3 in a graph, but 1 or 2-cycles can exist in a multigraph. A walk (respectively, trail, path) from u to v is called a (u, v)-walk (respectively, (u, v)-trail, (u, v)-path), and is sometimes written as $u \longrightarrow v$ for short. A walk can be also represented as a sequence of vertices when confusion does not occur.

A word span(ning) is used for the meaning of including all vertices, for example, we say spanning walks, spanning circuits, spanning subgraphs, etc.

A graph is called acyclic if it contains no cycles; unicyclic if it contains exactly one cycle (up to the choice of the initial vertex and the direction); and pancyclic if it contains cycles of every possible length (from 3 to the order of the graph).

The girth of a graph is the length of a shortest cycle in the graph; and the circumference, the length of a longest cycle. The girth and circumference of an acyclic graph are defined to be infinity ∞.

A spanning cycle (respectively, path) is called a Hamiltonian cycle (respectively, path), and a graph that contains a Hamiltonian cycle is called a Hamiltonian graph. A graph that contains a Hamiltonian path is traceable; and one that contains a Hamiltonian path for any given pair of (distinct) end vertices is a Hamiltonian connected graph.

A trail is called Eulerian if it uses all edges of a multigraph. A closed Eulerian trail is called an Eulerian circuit. A multigraph that contains an Eulerian trail is called traversable, and a (multi)graph that contains an Eulerian circuit is called an Eulerian (multi)graph.

Several paths are called (vertex-)independent (internally vertex-disjoint) if no two of them have any vertex in common, except the initial and terminal ones. Similarly, several paths are called edge-independent (edge-disjoint) if no two of them share any edge. The maximum number of independent (u, v)-paths is written as $\kappa'(u, v)$, and the maximum number of edge-independent (u, v)-paths is written as $\lambda'(u, v)$.

Theorem 1.3. *If a walk $u \longrightarrow v$ exists, then we have a path $u \longrightarrow v$ by taking a shortcut. Similarly, if a circuit passing an edge e exists, then we have a cycle passing e.*

(exercise) Find several walks, trails, paths, circuits, cycles in (multi)graphs in Figure 1-2.

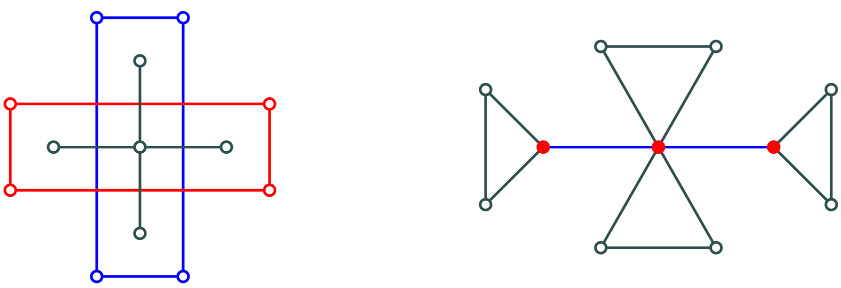


FIGURE 4

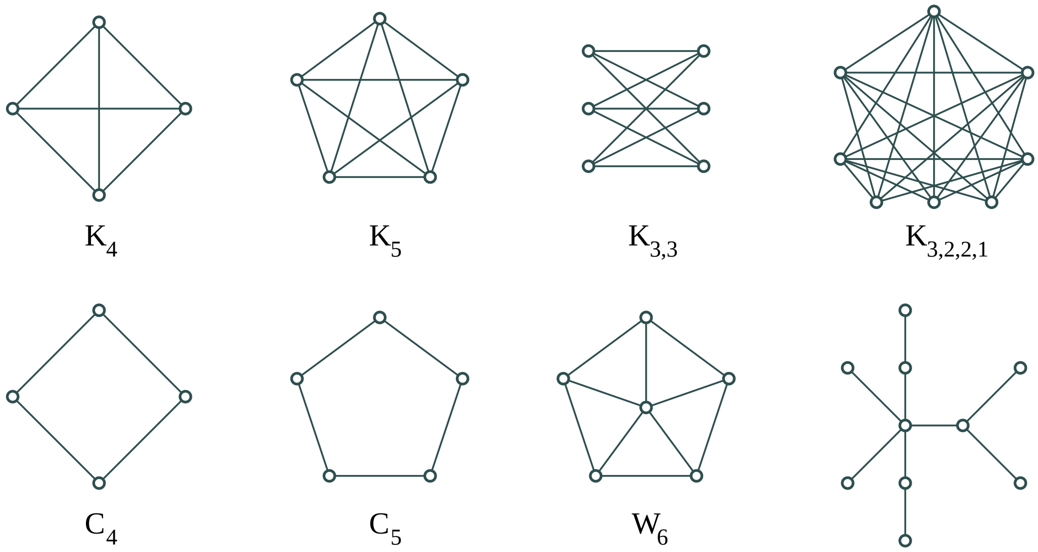


Figure 5

1.6 examples of graphs

If a (multi)graph G has a path between every two (distinct) vertices, then G is called connected. Otherwise, G is called disconnected. A maximal connected sub(multi)graph of G is called a (connected) component of G. The left graph in Figure 2 has two components, and the left graph in Figure 4 has three components: two rectangles and one cross.

A graph on n vertices where any distinct two vertices are adjacent is called the complete graph on n vertices, denoted by K_n. The size of K_n is $\binom{n}{2}$. K_n is $(n-1)$-regular.

For a graph $G = (V, E)$, if a partition $\{V_1, \ldots, V_s\}$ of V exists such that every cell V_i is independent (no two vertices in the same cell are adjacent) and any two vertices belonging to distinct cells V_i, V_j are adjacent, then G is called a complete s-partite (complete multipartite) graph. When $|V_i| = p_i$ $(i = 1, \ldots, s)$, write $G = K_{p_1,\ldots,p_s}$. A graph is called s-partite (multipartite) if it has a partition of V into at most s independent cells. Every subgraph of a complete s-partite graph is s-partite. For $s = 2, 3$, 2-partite and 3-partite are also called bipartite and tripartite, respectively. A graph is s-partite if and only if it is s-colorable. (See §1.9. Identify the cells of a graph with color classes.)

A cycle graph or circular graph of order n is a graph that consists of a single n-cycle, denoted by C_n. A path graph of order n is a graph that consists of a path of length $n-1$, denoted by P_n.

A wheel graph W_n is a graph on n vertices $(n \geq 4)$, formed by connecting a single vertex to all vertices of C_{n-1}.

A tree is a connected graph with no cycles, say, connected acyclic graph. A graph consisting of a single vertex is called a trivial tree. A tree of order n and diameter at most 2 is called an n-star, denoted by S_n, which is isomorphic to $K_{1,n-1}$. A tree on n

vertices of degree at most 2 is a path graph P_n. A tree can be characterized in many ways.

Theorem 1.4. *Let G be a graph with p vertices and q edges, then the following conditions are equivalent: (i) G is a tree. (ii) G is connected and $p = q + 1$. (iii) G is acyclic and $p = q + 1$. (iv) For any vertices u, v of G, there exists a unique path from u to v.*

A proof of this theorem is given in Chapter 4.

A forest is an acyclic graph. Every component of a forest is a tree, that is, a forest is a collection of trees.

(exercise) How many cycle graphs does K_n have? (especially $n = 4, 5$) (7,37)

(exercise) How many cycle graphs does W_n have?

(exercise) Find all trees on at most 8 vertices up to isomorphism.

1.7 connectivity

Let $G = (V, E)$ be a connected multigraph. A cut vertex of G is a vertex whose removal from G disconnects the remaining submultigraph. A cut set, or vertex cut or separating set, of G is a set of vertices whose removal from G disconnects the remaining submultigraph. A bridge of G is an edge whose removal from G disconnects the remaining submultigraph. A disconnecting set of G is a set of edges whose removal from G disconnects the remaining submultigraph. These terms are also used for a disconnected multigraph when their removals increase the number of components. An edge cut of G (with respect to $S, V - S$) is the set of all edges which have one end vertex in some proper nonempty vertex subset S and the other end vertex in $V - S$. All edges of K_3 form a disconnecting set but not an edge cut. Any two edges of K_3 form a minimum disconnecting set as well as an edge cut. An edge cut is necessarily a disconnecting set; and a minimum disconnecting set is necessarily an edge cut. A bond is a minimal (but not necessarily minimum) disconnecting set.

Let k be a positive integer. A connected multigraph G is said to be k-vertex-connected or k-connected if G has more than k vertices and any submultigraph formed by removing any $k-1$ vertices is still connected, or equivalently, if the size of any cut set of G is greater or equal to k. By definition, a k-connected multigraph is l-connected for every $1 \leq l \leq k$. The (vertex) connectivity $\kappa(G)$ of G is the minimum size of a cut set of G, that is the greatest number k such that G is k-connected, and therefore G is k-connected for and only for $1 \leq k \leq \kappa(G)$. (The above definitions are not clear when $G = K_n$ or G is a supermultigraph of K_n on n vertices because it has no cut sets; see below and §3.1.)

A connected multigraph G is said to be k-edge-connected if G has at least k non-loop edges and any submultigraph formed by removing any $k-1$ edges is still connected, or equivalently, if the size of any disconnecting set is greater or equal to k. By definition, a k-edge-connected multigraph is l-edge-connected for every $1 \leq l \leq k$. The edge connectivity $\lambda(G)$ of G is the minimum size of a disconnecting set of G, that is the greatest number k such that G is k-edge-connected, and therefore G is k-edge-connected for and only for $1 \leq k \leq \lambda(G)$. (The above definitions are not clear when G has only one vertex because it has no disconnecting sets; see below and §3.1.) One well-known result is that $\kappa(G) \leq \lambda(G) \leq \delta(G)$. (See §3.1.)

A (nontrivial) tree is 1-(edge-)connected, but not 2-(edge-)connected, thus it has vertex and edge connectivity 1. For $n \geq 2$, the complete graph K_n has edge connectivity $n-1$, and is defined to have vertex connectivity $n-1$. For $n = 1$, K_1, say a trivial tree (trivial graph) is defined to have vertex/edge connectivity 0. The vertex and edge connectivity of a disconnected multigraph are defined to be 0.

Theorem 1.5. *A connected multigraph has no bridges on its cycle (circuit). Furthermore,*

$$e \text{ is on a cycle} \iff e \text{ is not a bridge} \tag{8}$$

Proof. ($\Rightarrow$) Let G be a connected multigraph with a cycle C. Take any edge e on C and let $G' = G - e$. For any vertices u, v of G, we have a (u, v)-path w in G. If w does not pass through e, then w is also in G'. Otherwise, we have

$$w = u \dots xey \dots v. \tag{9}$$

Then we can go a long way round on C instead of xey, and have a new walk

$$w' = u \dots xe_1x_1 \dots x_{n-1}e_ny \dots v. \tag{10}$$

This does not pass e, and thus it is in G'. If it is not a path, then we can have a short-cut path $u \longrightarrow v$ in G'. □

($\Leftarrow$) If $e = xy$ is not a bridge, then we have a path $x \longrightarrow y$ not passing e. Adding e to this path, we have a cycle. □

(exercise) Find all cut vertices and bridges of the right graph in Figure 4.

(exercise) Find a graph with 3 cut vertices and 3 bridges.

(exercise) Find several graphs of connectivity ≤ 3 and edge-connectivity ≤ 3.

(exercise) Determine the connectivities and edge-connectivities of the (multi)graphs in Figure 5,6.

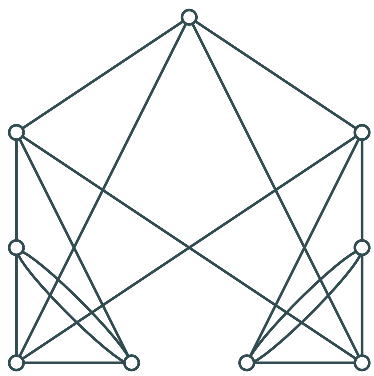


FIGURE 6

1.8 distance and diameter

Let G be a multigraph. The distance $d(u, v)$ between two (not necessarily distinct) vertices u and v of G is defined to be the length of a shortest path between them. By definition, $d(u, u) = 0$. When u and v are unreachable from each other, their distance is defined to be ∞. Thus, if two vertices belong to distinct components, their distance is ∞. The distance $d(u, v)$ satisfies the following axioms of distance.

$$\begin{aligned} &d(u,v) \geq 0 \\ &d(u,v) = 0 \iff u = v \\ &d(u,v) = d(v,u) \\ &d(u,v) \leq d(u,x) + d(x,v) \end{aligned} \tag{11}$$

The eccentricity $\epsilon(v)$ of a vertex v is the maximum distance between v and any other vertex. If G is not connected, $\epsilon(v) = \infty$ for all vertices v. The diameter $\mathrm{diam}(G)$ of G is the maximum distance between any two vertices, i.e., the maximum eccentricity over all vertices of G. The radius $\mathrm{rad}(G)$ is the minimum eccentricity over all vertices. For disconnected G, $\mathrm{diam}(G) = \mathrm{rad}(G) = \infty$. Vertices of maximum eccentricity (= $\mathrm{diam}(G)$) are called peripheral vertices. Vertices of minimum eccentricity (= $\mathrm{rad}(G)$) form the center. A tree has 1 or 2 center vertices. For two vertices u, v of maximum distance, taking a center vertex x, we have

$$\begin{aligned} \mathrm{diam}(G) &= d(u,v) \leq d(u,x) + d(x,v) \leq \mathrm{rad}(G) + \mathrm{rad}(G). \\ \therefore\ & \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G). \end{aligned} \tag{12}$$

The Wiener index of a vertex v of G, denoted by $W(v)$ is the sum of distances between v and all others. The Wiener index of G, denoted by $W(G)$, is the sum of distances between all unordered pairs of distinct vertices. The Wiener polynomial of G is defined to be the sum

$$W(G; q) = \sum_{\{u,v\}} q^{d(u,v)} \tag{13}$$

over the same set of pairs as before.

For sub(multi)graphs H_1, H_2 of G, the distance $d(H_1, H_2)$ between them is defined to be the length of a shortest path between a vertex of H_1 and a vertex of H_2.

A k-spanner of a graph G is a spanning subgraph, H, in which every two vertices are at most k-times as far apart on H than on G. The number k is called the dilation. k-spanners are used for studying geometric network optimization.

The k-th power G^k of a graph G is a supergraph of G formed from G by adding an edge between every nonadjacent pair of vertices whose distance is at most k. The second power G^2 is also called the square of G, the third power G^3 is called the cube of G, etc.

(exercise) Show (11).

(exercise) Determine the diameters of the graphs: the center of Figure 1, the center and right of Figure 2, and the right of Figure 4. (ans) 3,2,2,4.

(exercise) Determine the centers and the peripheral vertices of those graphs.

(exercise) Calculate the Wiener polynomials of P_n and C_n.

1.9 coloring and labeling

Graph coloring is an assignment of labels named “colors” to elements of graphs under certain restrictions. For a graph G, a (vertex) coloring of G, one kind of graph coloring, is to assign colors to the vertices so that adjacent vertices have distinct colors. A k-coloring of G is a coloring of G with at most k colors. G is k-colorable if G has a k-coloring. The chromatic number $\chi(G)$ of G is the smallest k for which G has a k-coloring. If $\chi(G) = k$, then G is called a k-chromatic graph. A minimum coloring of G is a coloring of G with $\chi(G)$ colors. It is well known that K_k and $K_{p_1,\ldots,p_k}$ are k-chromatic graphs.

Given a coloring of G, a color class of G is a set of vertices given the same color. G is k-critical if $\chi(G) = k$ but every proper subgraph of G has a smaller chromatic number. An odd cycle is 3-critical, and K_k is k-critical. G is critical if it is k-critical for some k.

The above definitions are naturally extended to multigraphs without loops, where multi-edges have no more effect than a single edge on vertex colorings. But a multigraph containing a loop vv has no vertex coloring, because by definition, v can not be colored.

An edge coloring of a multigraph G, another kind of graph coloring, is to assign colors to the edges so that adjacent edges have distinct colors. A k-edge-coloring of G is an edge coloring of G with at most k colors. G is k-edge-colorable if G has a k-edge-coloring. The chromatic index, or edge chromatic number, $\chi'(G)$ of G is the smallest k for which G has a k-edge-coloring. If $\chi'(G) = k$, then G is called a k-edge-chromatic graph. A minimum edge coloring of G is an edge coloring of G with $\chi'(G)$ colors. Different from vertex colorings, multi-edges and loops have a considerable effect on edge colorings.

Graph labeling is a generalization of graph coloring, referring to vertex labelings or edge labelings. Given a graph $G = (V, E)$, a vertex labeling is simply a function from V to a set of labels, usually represented by integers or sometimes by real numbers. A graph with such a function defined is called vertex-labeled. Similarly, an edge labeling is a function from E to a set of labels, then the graph is called edge-labeled. These definitions are also extended to multigraphs, without now, any restrictions.

When used without qualification, the term labeled graph generally refers to a vertex-labeled graph with all labels distinct. Such a graph may equivalently be labeled by the consecutive integers $\{1, 2, \ldots, n\}$, where n is the order of the graph.

(exercise) Find minimum colorings and minimum edge colorings of the graphs in Figure 5, and the graphs (1),(2) in Figure 16.

(exercise) For every $k = 1, \ldots, 5$, find several graphs with a chromatic number/index of k.

(exercise) Find several critical graphs.

1.10 multigraphs and matrices

Let G be a multigraph on $v_1, \ldots, v_p$ with edges $e_1, \ldots, e_q$. The adjacency matrix $A = (a_{ij})$ $(i, j = 1, \ldots, p)$ of G is defined as follows.

$$a_{ij} = (\text{the number of edges } v_i v_j) \tag{14}$$

The incidence matrix $M = (m_{ij})$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$ of G is defined by

$$m_{ij} = \begin{cases} 1 & (e_j \text{ is incident to } v_i) \\ 0 & (\text{otherwise}) \end{cases} \tag{15}$$

Each of the above two matrices of G can uniquely reconstruct G up to isomorphism.

The Laplacian matrix $L = (l_{ij})$ $(i, j = 1, \ldots, p)$ of G, sometimes called admittance matrix, Kirchhoff matrix or discrete Laplacian, is defined as $L = D - A$, where $D = (d_{ij})$ is the degree matrix of G, a diagonal $p \times p$ matrix indicating the degree of v_i by d_{ii}, and A is the adjacency matrix of G. The Laplacian matrix is used to calculate the number of spanning trees of a given multigraph without loops.

By definition, the adjacency/Laplacian matrix of a multigraph is always symmetric. Conversely, any symmetric matrix with nonnegative integer entries is represented as an adjacency matrix of some multigraph. We have

Theorem 1.6. *Let A be the adjacency matrix of a multigraph G. Let $A^n = \left(a_{ij}^{(n)}\right)$, then*

$$a_{ij}^{(n)} = (\textit{the number of walks } v_i \longrightarrow v_j \textit{ of length } n). \tag{16}$$

Proof. By induction on n. It is valid for $n = 1$ because a walk of length 1 corresponds to an edge. Suppose it is valid for $n - 1$. As $A^n = AA^{n-1}$,

$$\begin{aligned} a_{ij}^{(n)} &= \textstyle\sum_{k=1}^{p} a_{ik} a_{kj}^{(n-1)} \\ &= \textstyle\sum_{k=1}^{p} (\text{the number of walks } v_i \longrightarrow v_k \text{ of length } 1) \\ &\qquad \times (\text{the number of walks } v_k \longrightarrow v_j \text{ of length } (n-1)) \\ &= (\text{the number of walks } v_i \longrightarrow v_j \text{ of length } n). \end{aligned} \tag{17}$$

Thus the proposition is valid for n, and the induction is completed. □

Theorem 1.7. *Let G be a multigraph on p vertices, A the adjacency matrix of G, and E_p the identity matrix of degree p, then*

$$G \textit{ is connected.} \iff E_p + A + A^2 + \cdots + A^{p-1} \textit{ has no 0 entries.} \tag{18}$$

Proof.

$$\begin{aligned} & G \text{ is connected} \iff \text{a path } u \longrightarrow v \text{ exists for any } u, v \\ & \iff \text{a } (u, v)\text{-walk of length } \le p - 1 \text{ exists for any } u, v \\ & \iff E_p + A + A^2 + \cdots + A^{p-1} \text{ has no 0 entries.} \end{aligned} \tag{19}$$

□

The definitions of the above-mentioned three matrices of a multigraph G depend on the ordering of the vertices or edges of G. For the adjacency matrix or the Laplacian matrix, however, several properties such as the characteristic polynomial, eigenvalues, determinant, and minors (up to signs) are uniquely determined irrespective of the order of the vertices. Let

$$E_p = \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_p \end{pmatrix}. \tag{20}$$

Then for a permutation τ of p letters, let

$$P_\tau = \begin{pmatrix} \mathbf{e}_{\tau(1)} \\ \vdots \\ \mathbf{e}_{\tau(p)} \end{pmatrix} \tag{21}$$

be the permutation matrix of τ. We can confirm the following properties:

$$\begin{aligned} &P_\sigma P_\tau = P_{\tau\circ\sigma} \\ &P_\tau^{-1} = P_{\tau^{-1}} = P_\tau^T. \end{aligned} \tag{22}$$

When one changes the order of the vertices of G, the matrices A, L of G are also changed to $A' = P_\tau A P_\tau^T$, $L' = P_\tau L P_\tau^T$. Hence we have, for example, $|A'| = |A|$, and $\Phi_{A'}(t) = |tE_p - A'| = |P_\tau(tE_p - A)P_\tau^T| = |tE_p - A| = \Phi_A(t)$, which, the characteristic polynomial of the adjacency matrix, we call the characteristic polynomial of G denoted by $\Phi_G(t)$, and its roots are called the eigenvalues of G. Since A is symmetric, all eigenvalues of G are real. The (multi)set of all eigenvalues of G is called the spectrum of G. A nonzero vector $\mathbf{x}$ that satisfies the eigenequation $A\mathbf{x} = \lambda\mathbf{x}$ for some eigenvalue λ is called an eigenvector of A (or G) associated with λ, which depends on the order of the vertices.

Anyway, it is recommended to order the vertices/edges in every component to simplify the matrices. Let $G_1, \dots, G_s$ be the components of G, and $v_1, \dots, v_p$; $e_1, \dots, e_q$ be the vertices and edges, respectively, ordered from G_1 to G_s. Then the adjacency, incidence and Laplacian matrices of G are written as

$$\begin{aligned} &A = \begin{pmatrix} A_1 & O & \dots & O \\ O & A_2 & \dots & O \\ \dots & \dots & \dots & \dots \\ O & \dots & O & A_s \end{pmatrix} \qquad M = \begin{pmatrix} M_1 & O & \dots & O \\ O & M_2 & \dots & O \\ \dots & \dots & \dots & \dots \\ O & \dots & O & M_s \end{pmatrix} \\ &L = \begin{pmatrix} L_1 & O & \dots & O \\ O & L_2 & \dots & O \\ \dots & \dots & \dots & \dots \\ O & \dots & O & L_s \end{pmatrix}, \end{aligned} \tag{23}$$

where A_i, M_i, L_i are the adjacency, incidence and Laplacian matrices of G_i, respectively. By this expression, letting the order of G_i be $p(i)$, we have

$$\begin{aligned} \Phi_G(t) &= \Phi_A(t) = |tE_p - A| = \prod_{i=1}^{s} |tE_{p(i)} - A_i| \\ &= \prod_{i=1}^{s} \Phi_{A_i}(t) = \prod_{i=1}^{s} \Phi_{G_i}(t). \end{aligned} \tag{24}$$

(exercise) Write the adjacency/incidence/Laplacian matrices of the following multigraph G.

(exercise) Calculate the characteristic polynomial of G and find several eigenvalues.

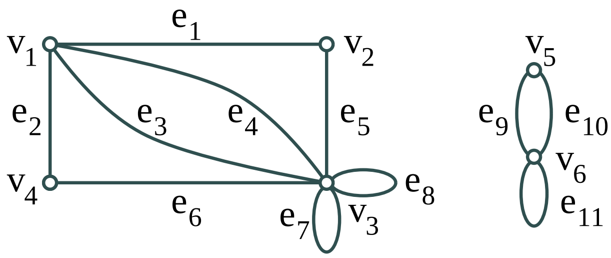


FIGURE 7

(ans)

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{25}$$
$$\begin{pmatrix} 4 & -1 & -2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -2 & -1 & 6 & -1 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -2 & 3 \end{pmatrix}$$

(ans) $\Phi_G(t) = t(t^3 - 2t^2 - 8t - 4)(t^2 - t - 4)$.

(exercise) Calculate the characteristic polynomials of K_n and $K_{m,n}$, and find all eigenvalues. (ans) $\Phi_{K_n}(t) = (t - n + 1)(t + 1)^{n-1}$, $\Phi_{K_{m,n}}(t) = t^{m+n-2}(t^2 - mn)$.

(exercise) Let $\Phi_{P_n} = \Phi_n$, then show that $\Phi_n = t\Phi_{n-1} - \Phi_{n-2}$.

(exercise) Calculate the characteristic polynomials of C_n and W_n for $n \leq 6$.

(exercise) Show that every n-regular graph has an eigenvalue n.

(exercise) Let $G = K_{p,\ldots,p}$ be a complete s-partite graph on n vertices. Show that $\Phi_G(t) = t^{n-s}(t + p)^{s-1}(t - n + p)$.

Theorem 1.8. *Let $G = K_{p_1,\ldots,p_s}$ be a complete s-partite graph on n vertices. We have*

$$\Phi_G(t) = t^{n-s} \begin{vmatrix} t & -p_2 & -p_3 & \ldots & -p_s \\ -p_1 & t & -p_3 & \ldots & -p_s \\ -p_1 & -p_2 & t & \ldots & -p_s \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -p_1 & -p_2 & -p_3 & \ldots & t \end{vmatrix}, \tag{26}$$

where for $s = 1$, G is regarded as an independent set.

Proof. For $G = K_{p_1,\dots,p_s}$, denote $\Phi_{p_1,\dots,p_s} = \Phi_G(t)$. Denote by 1_{pq} a $p \times q$ matrix consisting of only 1's. By definition, $\Phi_{p_1,\dots,p_s}$ is expressed by the determinant:

$$\begin{vmatrix} tE_{p_1} & -1_{p_1p_2} & -1_{p_1p_3} & \dots & -1_{p_1p_s} \\ -1_{p_2p_1} & tE_{p_2} & -1_{p_2p_3} & \dots & -1_{p_2p_s} \\ -1_{p_3p_1} & -1_{p_3p_2} & tE_{p_3} & \dots & -1_{p_3p_s} \\ \dots & \dots & \dots & \dots & \dots \\ -1_{p_sp_1} & -1_{p_sp_2} & -1_{p_sp_3} & \dots & tE_{p_s} \end{vmatrix}. \tag{27}$$

Subtract the second row from the first row, and again subtract the second column from the first column. By the cofactor expansion along the first row (or column), we have the following recursive formula:

$$\Phi_{p_1,\dots,p_s} = 2t\Phi_{p_1-1,p_2,\dots,p_s} - t^2\Phi_{p_1-2,p_2,\dots,p_s} \quad (p_1 \geq 2), \tag{28}$$

where we set $\Phi_{0,p_2,\dots,p_s} = \Phi_{p_2,\dots,p_s}$. We can confirm that (26) satisfies (28) and the initial conditions: $\Phi_{1,\dots,1} = \Phi_{K_n}$ and $\Phi_p = t^p$. □

2 Eulerian/Hamiltonian multigraphs

2.1 Eulerian multigraphs

There is a simple criterion to determine whether a finite (multi)graph is Eulerian/traversable or not, it says:

Theorem 2.1. *Let G be a finite connected (multi)graph, then*

$$G \text{ is an Eulerian (multi)graph} \iff G \text{ has no odd vertices} \tag{29}$$

$$G \text{ is traversable} \iff G \text{ has 0 or 2 odd vertices} \tag{30}$$

Proof. (29) ($\Rightarrow$) Let G be an Eulerian multigraph, and w be its Eulerian circuit. Take an arbitrary vertex v and consider its degree. If v is not the initial (= terminal) vertex of w, then the Eulerian circuit w consumes exactly two edges (every loop is counted twice) incident to v as it passes through v, until it exhaust all edges incident to v. If v is the initial (= terminal) vertex, w consumes first one edge, next two edges as w passes, finally one edge. Hence v has even degree. Therefore G has no odd vertices. □

($\Leftarrow$) Let G be connected and has no odd vertices. Let w be one of the longest closed trails in G. We show w is an Eulerian circuit by reduction to absurdity. Suppose w is not an Eulerian circuit. Then there exist edges not on w, and some of them, say e, is incident to some vertex on w. For, if not, a vertex on w can not reach any vertices not on w by a path, but G is connected, thus w spans G, and so e exists, contradiction. Hence suppose e not on w is incident to u on w. Now we develop a trail w' from ue not using edges on w, as long as possible. Then w' returns to u, because the number of edges incident to each vertex is even, and the number of already used ones by w is also even, hence even edges incident to each vertex remain unused, and when w' comes to v, odd edges incident to v remain, and w' can go ahead until it returns to u. Thus we have $w' = ue \dots e'u$. We compose $w = ue_1 \dots e_n u$ and w' as $\tilde{w} = ue_1 \dots e_n ue \dots e'u$, which is longer than w, contradiction. □

(30) ($\Rightarrow$) Let G be traversable and $w = u \dots v$ (possibly $u = v$) be an Eulerian trail. Clearly, we have an Eulerian multigraph G' by adding an edge uv. By (29), G' has no odd vertices, and so G has no odd vertices except u, v. If $u = v$, all vertices are even. □

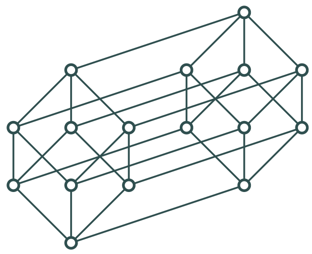
($\Leftarrow$) Let G be connected and satisfy the right hand side of (30). If G has no odd vertices, by (29), G has an Eulerian circuit. If G has exactly two odd vertices u, v, by adding an edge uv, we have G' without odd vertices. Then by (29), G' has an Eulerian circuit, and therefore G has an Eulerian trail. □

A general (not necessarily connected) Eulerian multigraph is composed of a connected Eulerian component and isolated vertices. Similarly, a general traversable multigraph is composed of a connected traversable component and isolated vertices. Therefore we have the following.

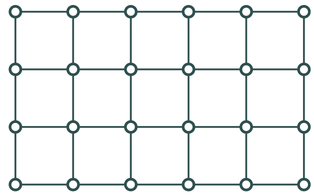
Theorem 2.1'. *Let G be a finite (multi)graph. (i) G is an Eulerian (multi)graph if and only if G is connected except isolated vertices, and has no odd vertices. (ii) G is traversable if and only if G is connected except isolated vertices, and has 0 or 2 odd vertices.*

(exercise) Find all Eulerian graphs of order ≤ 6, and also several connected Eulerian multigraphs.

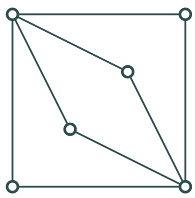
(exercise) Find all traversable graphs of order ≤ 6, and also several connected traversable multigraphs.



an Eulerian and Hamiltonian graph



a Hamiltonian graph



an Eulerian graph

FIGURE 8

2.2 Hamiltonian graphs

In general, no good criteria are known to determine whether a graph is Hamiltonian or not. It is a difficult open problem in graph theory.

(exercise) Let $L_{m,n}$ denote a lattice graph consisting of a rectangular array of $m \times n$ vertices, where every horizontally or vertically "adjacent" pair of vertices is connected by an edge. The upper right graph in Figure 8 is $L_{4,6}$. Show that $L_{m,n}$ is Hamiltonian if and only if mn is even.

3 Connectivity 3.1 vertex/edge connectivity

In Section 1.7, we defined the vertex/edge connectivity of a multigraph. That is the minimum size of a cut/disconnecting set of a multigraph. However, the trivial graph K_1 is specially treated, the vertex/edge connectivity of K_1 is defined to be 0, although it is a connected graph. A looped vertex, say, a multigraph on one vertex with several loops, is also defined to have vertex/edge connectivity 0. After this definition, edge connectivity is clearly defined, whereas vertex connectivity has something cumbersome yet. First of all, note that vertex/edge connectivity is independent of whether loops exist or not, and vertex connectivity is independent of whether the multiplicity of an edge is 1 or not. Next, the complete graph K_n can not be disconnected by removing vertices; the vertex connectivity of K_n is defined to be $n-1$. Every supermultigraph of K_n on n vertices is also defined to have vertex connectivity $n-1$.

Theorem 3.1. *Let G be a connected multigraph. An edge cut of G is necessarily a disconnecting set of G; and a minimum disconnecting set of G is necessarily an edge cut of G.*

Proof. Let F be an edge cut of G with respect to S, $V-S$. Then it is clear that F is a disconnecting set of G. Let F be a minimum disconnecting set of G. Then $G-F$ consists of exactly 2 components, say, H, H', because of the minimality of F. Hence F is an edge cut of G with respect to $V(H)$, $V(H')$. □

For $G = K_n$, consider the edge connectivity of G. Let F be a minimum disconnecting set of G. By this theorem, F is an edge cut of G with respect to some S, S'. Then we can see easily that the minimum value of $|F|$ is $n-1$, say, $\lambda(G) = n-1$.

Theorem 3.2. *(H. Whitney) Let G be a finite multigraph, then $\kappa(G) \le \lambda(G) \le \delta(G)$.*

Proof. If G is disconnected, the inequality is clear. Hence let $G = (V, E)$ be a finite connected multigraph. If $\delta(G) = \deg(v)$, we can disconnect G by removing all edges incident to v. Thus $\lambda(G) \le \delta(G)$. Next we prove $\kappa(G) \le \lambda(G)$. For simplicity, we may assume that G has no loops. Also, we assume that G has no multi-edges, because multi-edges make only the edge connectivity increase. If G is a complete graph, then $\kappa(G) = \lambda(G)$. Hence, it suffices to show the inequality for a finite connected graph G which is not complete. Let F be a minimum disconnecting set of s edges. By Theorem 3.1, F is an edge cut of G and $G-F$ has exactly 2 components H, H'. Let $S = V(H)$, $S' = V(H') = V - S$. Let V' be the set of end vertices of all edges in F. Let $T = S \cap V'$ and $T' = S' \cap V'$. If $T \neq S$, then we remove all elements of T from G to remove all of F, this can be done by deletion of at most s vertices, in order to disconnect G. The case $T' \neq S'$ is treated similarly.

Thus, consider the case that $T = S$, $|T| = m$, and $T' = S'$, $|T'| = n$. Suppose there exist nonadjacent vertices x, x' such that $x \in T$ and $x' \in T'$. Then removing at most s vertices except x, x' to remove all of F, we have a disconnected graph including x and x'. Next suppose any vertices $x \in T$ and $x' \in T'$ are adjacent. Then there exist vertices $x, y \in T$ or $x', y' \in T'$ such that x, y or x', y' are not adjacent, because G is not complete. Let $x, y \in T$ be nonadjacent vertices. Remove all vertices of G except

x and y, then we have a disconnected graph of independent vertices x, y. The number of removed vertices are $m + n - 2$, which is less than $s = mn$. $\square$

3.2 Menger's theorem

Menger's theorem is one of the most important facts about connectivity in graphs, which combines the local connectivity and the number of independent paths. Let u, v be vertices of a graph G. The local connectivity $\kappa(u, v)$ between u and v is the minimum number of vertices (except u and v) that need to be removed to separate u from v (to kill all (u, v)-paths). Clearly $\kappa(u, v) = \kappa(v, u)$, say, local connectivity is symmetric. Similarly, the local edge-connectivity $\lambda(u, v)$ is the minimum number of edges to remove to separate u from v, which is also symmetric. Then Menger's theorem says:

Theorem 3.3. *(K. Menger, 1927) Let G be a finite graph, and u, v be distinct vertices of G, then*

$$\begin{aligned} &\kappa(u, v) = \kappa'(u, v) \qquad (u, v \text{ are nonadjacent}) \\ &\lambda(u, v) = \lambda'(u, v). \end{aligned} \tag{31}$$

To prove this, we show another version of Menger's theorem (Theorem 3.4). First we introduce several notions. Let G be a multigraph, and let A, B be two sets of vertices of G. An A-B path is a path from a vertex in A to a vertex in B. An A-B path is proper if inner vertices are not in A nor B. Every A-B path contains a proper A-B path as a part.

A set of vertices X separates A from B (X is a separating set between A and B) if every (proper) A-B path meets X. (This definition is independent of whether "proper" is used or not.) If $|X| = s$, then X is an s-separating set between A and B. Let $\kappa(G, A, B)$ denote the minimum size of a separating set between A and B in G.

A set of edges F disconnects A from B (F is a disconnecting set between A and B) if every (proper) A-B path meets F. If $|F| = s$, then F is an s-disconnecting set between A and B. Let $\lambda(G, A, B)$ denote the minimum size of a disconnecting set between A and B in G.

Several A-B paths are (vertex-)disjoint if no two of them share any vertex. Let $\kappa^*(G, A, B)$ denote the maximum number of disjoint (proper) A-B paths in G. Several A-B paths are edge-disjoint if no two of them share any edge. Let $\lambda^*(G, A, B)$ denote the maximum number of edge-disjoint (proper) A-B paths in G.

Let H be a sub(multi)graph of G. An H-path is a nontrivial path which meets H exactly in its ends, say, a nontrivial path not meeting H except that its ends meet H.

Theorem 3.4. *Let G be a finite graph, and A, B be sets of vertices of G, then*

$$\kappa(G, A, B) = \kappa^*(G, A, B), \qquad \lambda(G, A, B) = \lambda^*(G, A, B). \tag{32}$$

Here, for the first equality, A and B may have an intersection, while for the second equality, suppose $A \cap B = \varnothing$.

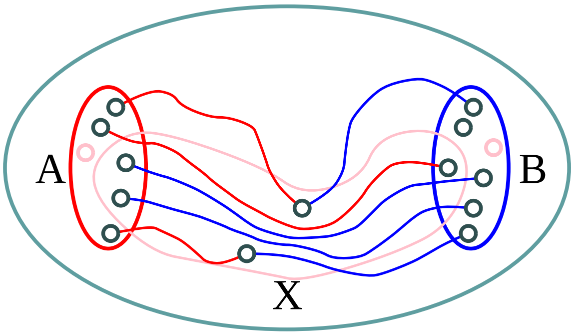


FIGURE 9

Proof. (The first equality) If s disjoint A-B paths exist, then we can not choose less than s vertices which separate A from B. Hence we have

$$\kappa(G, A, B) \geq \kappa^*(G, A, B). \tag{33}$$

Therefore we prove the equivalent propositions: the first equality of (32) ((32)-1, for short), or (*): for a given graph G, if $\kappa(G, A, B) = s$, then there exist s disjoint A-B paths in G, by induction on the partial order over graphs defined by (1). If G is an independent set, then (32)-1 holds. Let G be a graph and suppose (32)-1 holds for every graph less than G. Let $\kappa(G, A, B) = s$ and X be an s-separating set between A and B.

(i) First of all, we consider the case $A \cap B \neq \varnothing$. Let $c \in A \cap B$, then c is an A-B path, and thus $c \in X$. Hence $\kappa(G, A, B) = \kappa(G - c, A - c, B - c) + 1$. By the induction hypothesis, $\kappa(G - c, A - c, B - c) = \kappa^*(G - c, A - c, B - c) = s - 1$. Hence we have s disjoint A-B paths in G.

(ii) Next, we consider the case $A \cap B = \varnothing$. Further we suppose that $X \neq A, B$. By the minimality of X, $A - X, B - X \neq \varnothing$. Let $b \in B - X$. We have $\kappa(G, A, X) = s$, because $|X| = s$ implies $\kappa(G, A, X) \leq s$, and if $\kappa(G, A, X) = s' < s$, then as every A-B path meets X, $\kappa(G, A, B) \leq s'$, contradiction.

If $A \cap X \neq \varnothing$, then this case is reduced to (i), and we have $\kappa(G, A, X) = \kappa^*(G, A, X) = s$. Hence we suppose that $A \cap X = \varnothing$. Noting that no proper A-X path passes through b, $\kappa(G - b, A, X) = \kappa(G, A, X) = s$. By the induction hypothesis, $\kappa(G - b, A, X) = \kappa^*(G - b, A, X) = s$, and therefore, using (33) with B replaced by X, $\kappa^*(G, A, X) = s$. In a similar manner, we have $\kappa^*(G, X, B) = s$.

In this stage, we have s disjoint proper A-X paths and s disjoint proper X-B paths, but moreover, all of them are disjoint except vertices X, because if not, we have an A-B path not meeting X. Now just connecting them, we have s disjoint A-B paths as desired (Figure 9).

(iii) Lastly, we suppose that $A \cap B = \varnothing$, and $X = A$ or B. We may assume that a proper A-B path w exists. (Otherwise, A is already separated from B and $\kappa(G, A, B) = 0 = \kappa^*(G, A, B)$.) Take any edge $e = xy$ in w, where x and y appear in w in this order, then we have $x \notin B$ and $y \notin A$. If $\kappa(G - e, A, B) = s$, then by the induction hypothesis, $\kappa^*(G, A, B) \geq \kappa^*(G - e, A, B) = s$, and thus (*) holds for G. If $\kappa(G - e, A, B) = s - 1 \geq 1$ and X' is an $(s - 1)$-separating set between A and B in $G - e$, then we can get an s-separating set $X = X' \cup \{x\}$ or $X' \cup \{y\}$ between A and

B in G, which is not equal to A nor B; this case is reduced to the previous case (ii). The case $s-1=0$ is trivial. The case $\kappa(G-e,A,B) \le s-2$ is clearly impossible. $\square$

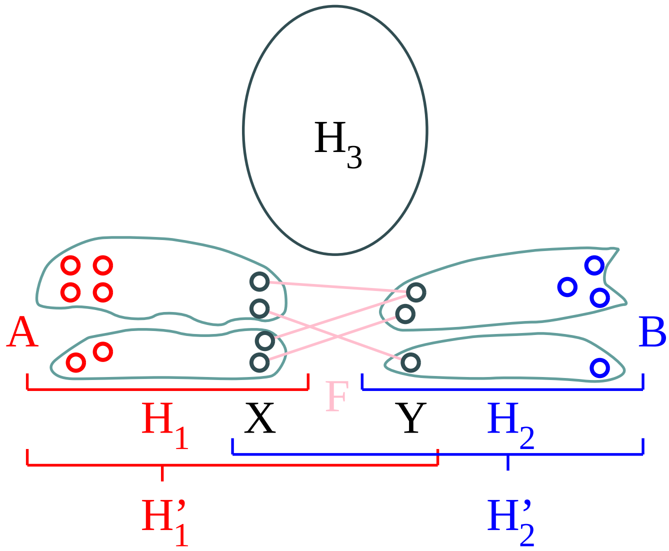


FIGURE 10

Proof. (The second equality) If s edge-disjoint A-B paths exist, then we can not choose less than s edges which disconnect A from B. Thus we have $\lambda(G,A,B) \ge \lambda^*(G,A,B)$. Hence, like the first proof, we show the equivalent propositions: the second equality of (32) ((32)-2, for short), or (*): for a graph G, if $\lambda(G,A,B)=s$, then there exist s edge-disjoint A-B paths in G, by induction on the partial order over graphs. If G is an independent set, then (32)-2 holds. Let G be a graph and suppose (32)-2 holds for every graph less than G. Let $\lambda(G,A,B)=s$ and F be an s-disconnecting set between A and B.

Consider the components of $G-F$, and let H_1 (respectively, H_2) be the union of all components containing some elements of A (respectively, B). Let H_3 be the union of all components containing no elements of $A \cup B$. By the minimality of F, every edge in F connects some vertex of H_1 and some vertex of H_2. Let V' be the set of end vertices of all edges in F, and define $X = V' \cap V(H_1)$ and $Y = V' \cap V(H_2)$. Let H_1' and H_2' be the graphs made by adding all edges in F to H_1 and H_2, respectively (Figure 10). If $\lambda(H_1',A,Y) < s$ or $\lambda(H_2',X,B) < s$, then $\lambda(G,A,B) < s$, contradiction. Hence, noting the s-disconnecting set F, we have $\lambda(H_1',A,Y) = \lambda(H_2',X,B) = s$.

Suppose $H_1', H_2' < G$. By the induction hypothesis, we have s edge-disjoint A-Y paths in H_1' and s edge-disjoint X-B paths in H_2'. Connecting these paths putting F between them, we have s edge-disjoint A-B paths in G.

Suppose $H_1' = G$ or $H_2' = G$. By the minimality of F, we have $H_1' = G \Rightarrow B = Y$ and $H_2' = G \Rightarrow A = X$. Thus we suppose $A = X$ or $B = Y$. If the maximum length of a proper A-B path ≤ 2, then G's structure is very restricted and we can confirm (*) holds for G (Figure 11). If the maximum length of a proper A-B path ≥ 3, then there exists an edge $e = xy$ such that $x, y \notin A \cup B$. If $\lambda(G-e,A,B) = s$, then by the induction hypothesis, $\lambda^*(G,A,B) \ge \lambda^*(G-e,A,B) = s$, and (*) holds for G.

If $\lambda(G-e,A,B)=s-1$ and F' is an $(s-1)$-disconnecting set between A and B in $G-e$, then $F=F'\cup\{e\}$ is an s-disconnecting set between A and B in G, and this case is reduced to the case that $A\neq X$ and $B\neq Y$, say, $H_1', H_2' < G$. $\square$

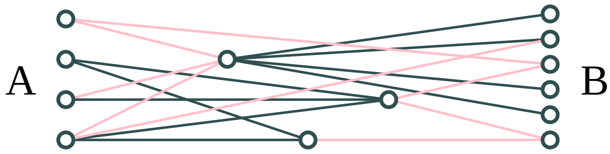


FIGURE 11

Theorem 3.4 holds also for finite multigraphs G; the proofs are valid for this case without correction. The second equality of (32) gives an essentially extended result for the multigraph case. The first equality, however, has no deeper meaning for the multigraph case, because multi-edges or loops make no difference for both sides of the equality.

Proof of Theorem 3.3. The second equality is a special case of the second one of Theorem 3.4. The first equality is shown as follows. Let G be a finite graph and u, v be distinct nonadjacent vertices of G. Let A and B be the sets of all neighbors of u and v, respectively, that is, $A=V(N(u))$, $B=V(N(v))$. By Theorem 3.4, we have $\kappa(u,v)=\kappa(G,A,B)=\kappa^*(G,A,B)=\kappa'(u,v)$. $\square$

Theorem 3.5. *(The global version of Menger's theorem) Let G be a connected multigraph. Let k be a positive integer. For (i), suppose that G is neither K_n nor any supermultigraph of K_n on n vertices, for any n. For (ii), suppose that G has at least two vertices.*
(i) G is k-connected if and only if it has k independent paths between any two distinct nonadjacent vertices.
(ii) G is k-edge-connected if and only if it has k edge-disjoint paths between any two distinct vertices.

Proof. (i) Let G be k-connected. Let u, v be arbitrary distinct nonadjacent vertices of G. Let $X \not\ni u, v$ be a minimum separating set between u and v. Then X is a cut set of G and so $|X|\geq k$. Hence $\kappa'(u,v)=\kappa(u,v)\geq k$, and therefore k independent (u,v)-paths exist.

Conversely, suppose that G has k independent paths between any two distinct nonadjacent vertices. Let X be an arbitrary cut set of G. Let u and v be vertices of distinct components of $G-X$. Then u and v are clearly distinct, nonadjacent and X is a separating set between them. Hence $|X|\geq\kappa(u,v)=\kappa'(u,v)\geq k$, say, $|X|\geq k$, and therefore G is k-connected. $\square$

Proof. (ii) is proved in a similar manner to the proof of (i). $\square$

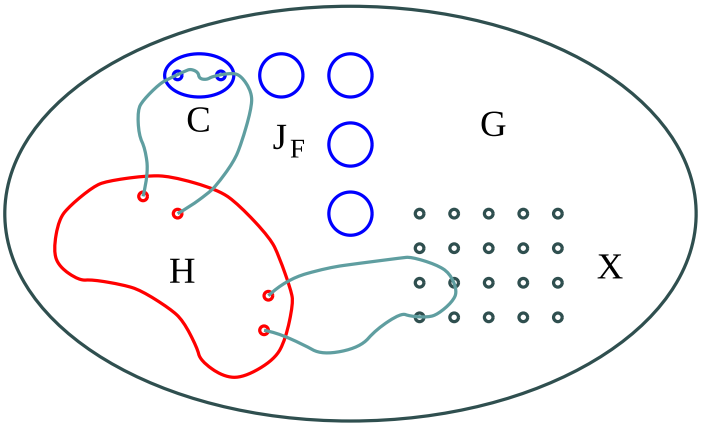


FIGURE 12

(exercise) Find minimum cut sets and minimum disconnecting sets of several (multi) graphs.

(exercise) For some $L_{m,n}$ and some vertex sets A, B, find the following: (1) A minimum separating set between A and B. (2) A maximum set of disjoint A-B paths. (3) A minimum disconnecting set between A and B. (4) A maximum set of edge-disjoint A-B paths.

3.3 Mader's theorem

In this section, we introduce a deep result of Mader without a proof. Let G be a finite graph and H be an induced subgraph of G. Mader's theorem describes the maximum number of independent H-paths in G. On the analogy of Menger's theorem, we consider a kind of obstacles for H-paths. Choose a vertex set $X \subset V(G-H)$ and an edge set $F \subset E(G-H)$ not contained in H, such that every H-path meets X or F. For such X and F, the number of independent H-paths is at most $|X \cup F|$. For our purpose, we may omit all edges in F incident to some vertex in X, because a path with such an edge always meets X. Further, since H is an induced subgraph, the length of an H-path should be greater than 1. Hence we can choose X, $Y = V(G-H) - X$ and a graph $J_F = (Y, F)$ $(\le G[Y])$, such that every H-path meets X or F.

Now for a component C of J_F, let ∂C denote the set of all vertices of C adjacent to some vertex of $G - C - X$. An H-path avoiding X should meet F, and so at least two vertices in ∂C for some C. Thus, the quantity:

$$m_G(H) = \min \left(|X| + \sum_{C:\text{ components of } J_F} \left[\tfrac{1}{2}|\partial C|\right] \right) \tag{34}$$

is an upper bound of the number of independent H-paths, where the minimum is taken over all X and F described above.

Then Mader's theorem says:

Theorem 3.6. *(W. Mader, 1978) Let G be a finite graph and H be an induced subgraph of G. The maximum number of independent H-paths in G is $m_G(H)$.*

4 Trees 4.1 basics

Trees, connected acyclic graphs, form an important class of graphs. A (nontrivial) tree is 1-(edge-)connected, that connects the vertices with least edges. The number of trees of order n up to isomorphism, denoted by $t(n)$, increases rapidly and the first few values are

$$1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159. \tag{35}$$

However, no exact formula for $t(n)$ is known.

In this chapter, we study mainly about spanning trees from several viewpoints, starting with several basic properties of trees.

Lemma 4.1. *Let T be a tree with p vertices and q edges, then $p = q + 1$.*

Proof. By induction on p. For $p = 1$, it is valid. Suppose it is valid for $p - 1$. Let T be a tree with p vertices and q edges. Consider a longest path $u \longrightarrow v$ in T. Then $\deg(u) = \deg(v) = 1$, because if not, we have a longer path or a cycle. Hence $T - v$ is a tree with $p - 1$ vertices and $q - 1$ edges. By the induction hypothesis, we have $p - 1 = q - 1 + 1$. □

Theorem 1.4. *Let G be a graph with p vertices and q edges, then the following conditions are equivalent: (i) G is a tree. (ii) G is connected and $p = q + 1$. (iii) G is acyclic and $p = q + 1$. (iv) For any vertices u, v of G, there exists a unique path from u to v.*

Proof. (i) $\Longleftrightarrow$ (ii) : By Lemma 4.1, (i) $\Rightarrow$ (ii) is clear. We prove (i) $\Leftarrow$ (ii). Let G be a connected graph with p vertices and q edges such that $p = q + 1$. If G has a cycle, we can remove an edge on the cycle to have a connected graph G'. Repeating this process, we have a connected acyclic graph $G^{(s)}$, tree. Then $p = q - s + 1$. $\therefore$ $s = 0$. □

(i) $\Longleftrightarrow$ (iii) : It suffices to prove (i) $\Leftarrow$ (iii). Let G be an acyclic graph such that $p = q + 1$. Every component G_i of G is a tree, say $p_i = q_i + 1$. Hence $p = q + s$, where s is the number of components, and therefore $s = 1$. Hence G is a tree. □

(i) $\Rightarrow$ (iv) : Let G be a tree. Let u, v be arbitrary vertices of G. As G is connected, G has a (u, v)-path. We derive a contradiction from the assumption that there exist distinct two (u, v)-paths w and w' in G. Suppose w and w' share a common path from u to u_0, and the next vertices in w and w' are x and x' ($x \neq x'$), respectively. Let y be the first vertex after x in w, such that y is also in w'. (y appears after x' also in w'.) Then we have a cycle $u_0 x \longrightarrow y \longrightarrow x' u_0$, which is a contradiction. □

(i) $\Leftarrow$ (iv) Suppose (iv) is satisfied. Then G is connected. If G has a cycle C, then an edge $e = xy$ on C is not a bridge, and thus there exists an (x, y)-path in $G - e$. Hence we have two distinct (x, y)-paths in G, a contradiction. Thus G is acyclic, and therefore G is a tree. □

(exercise) Prove a tree is 2-colorable, equivalently bipartite.
(exercise) Let G be a forest consisting of s components. Determine $p - q$.

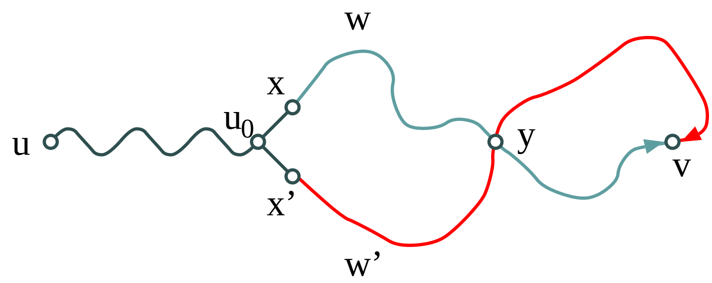


FIGURE 13

4.2 spanning trees and Kirchhoff's theorem

Let G be a multigraph. A spanning tree of G is a tree which is a spanning subgraph of G. In other words, a spanning tree of G is a tree formed by removing several edges from G. Obviously, if G is disconnected, then G has no spanning trees. If G is connected, then we can remove an edge on some cycle of G to get connected G', and repeating this process, we have a spanning tree of G. It is important to enumerate the number $t(G)$ of spanning trees of a connected multigraph G, and Kirchhoff's theorem (Kirchhoff's matrix tree theorem) is a surprising result that combines $t(G)$ with a determinant:

Theorem 4.1. *(Kirchhoff's theorem) Let G be a multigraph without loops. Then $t(G)$ is equal to the (i, j) cofactor of the Laplacian matrix of G for every i, j.*

To prove this, we need several theorems and lemmas.

Theorem 4.2. *(Cauchy–Binet formula) Let $m \leq n$ be positive integers. Let A be an $m \times n$ matrix and B an $n \times m$ matrix, then*

$$\det(AB) = \sum_{S} \det(A_S) \det\left(B^S\right), \tag{36}$$

where S runs over all m-element subsets of $[n] = \{1, 2, \ldots, n\}$, and for $S = \{j_1, \ldots, j_m\}$, write A_S for the $m \times m$ matrix consisting of the $j_1, \ldots, j_m$-th columns of A, and B^S for the $m \times m$ matrix consisting of the $j_1, \ldots, j_m$-th rows of B.

Proof. Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $B = (b_{ij})$. By multilinearity of determinants, we have

(37)
$$\begin{aligned}
\det(AB) &= \det\left(\sum_{j=1}^n b_{j1}\mathbf{a}_j \quad \ldots \quad \sum_{j=1}^n b_{jm}\mathbf{a}_j\right)\\
&= \sum_{\{j_1,\ldots,j_m\}\subset[n]} b_{j_1,1}\ldots b_{j_m,m}\det\left(\mathbf{a}_{j_1}\ldots\mathbf{a}_{j_m}\right)\\
&= \sum_{1\le i_1<\cdots<i_m\le n}\sum_{*}\operatorname{sgn}\begin{pmatrix} i_1 & \ldots & i_m\\ j_1 & \ldots & j_m\end{pmatrix} b_{j_1,1}\ldots b_{j_m,m}\det\left(\mathbf{a}_{i_1}\ldots\mathbf{a}_{i_m}\right)\\
&= \sum_{S\subset[n]}\det\left(B^S\right)\det\left(A_S\right).
\end{aligned}$$

Here, the summation $*$ runs over all permutations $j_1, \ldots, j_m$ of $i_1, \ldots, i_m$. □

Let G be a multigraph on p vertices, with q edges without loops. The oriented incidence matrix M of G is given by changing the sign of the second nonzero entry in each column of the ordinary incidence matrix. For example, the oriented incidence matrix of K_5 is shown below.

(38)
$$\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0\\
0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0\\
0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1
\end{pmatrix}$$

Let L be the Laplacian matrix of G and L_{ij} be the submatrix obtained by deleting the i-th row and the j-th column of L. Let F be the submatrix obtained by deleting the i-th row of M. Then we have

(39)
$$L = M\,{}^tM, \qquad L_{ii} = F\,{}^tF.$$

By the Cauchy–Binet formula,

(40)
$$\det L_{ii} = \det\left(F\,{}^tF\right) = \sum_S \det\left(F_S\right)\det\left({}^tF_S\right) = \sum_S \left(\det F_S\right)^2,$$

where S runs over all $(p-1)$-element subsets of $[q]$.

Lemma 4.2. *Let G be a multigraph on p vertices with $q = p-1$ edges without loops. Let F be a matrix obtained by removing a row from the oriented incidence matrix of G. Then $\det F = \pm 1$ if and only if G is a tree, and $\det F = 0$ if and only if G is not a tree.*

Proof. If G has a cycle, then one can see that $\det F = 0$. Otherwise G is a forest, and by $q = p-1$, G is a tree. We can prove $\det F = \pm 1$ if G is a tree, by induction on the number of vertices, but we omit the details. □

Proof of Theorem 4.1. By Lemma 4.2, in (40), if the edges corresponding to S form a spanning tree, then $(\det F_S)^2 = 1$, otherwise $\det F_S = 0$. Hence the determinant of L_{ii} counts exactly the number of spanning trees.

The next thing to prove is that, for the cofactors $\tilde{l}_{ij} = (-1)^{i+j} \det L_{ij}$ of L, they have the same value for all i, j. If G is disconnected, it is clear that $\det L_{ij} = 0$ by the expansion of $\det L_{ij}$ which is similar to (40). Thus we suppose that G is connected. On the one hand, L is singular because

$$L\mathbf{x}_0 = M\,{}^tM\mathbf{x}_0 = M\mathbf{0} = \mathbf{0} \tag{41}$$

for $\mathbf{x}_0 = {}^t(1, \ldots, 1)$. On the other hand, as G is connected, M includes a nonsingular $(p-1) \times (p-1)$ submatrix, and therefore the rank of the $p \times p$ matrix $L = M\,{}^tM$ is $p-1$. Hence the equation

$$L\mathbf{x} = \mathbf{0} \tag{42}$$

has a 1-dimensional solution space $\mathrm{span}\{\mathbf{x}_0\}$. For the adjugate matrix $\mathrm{adj}\, L = {}^t\left(\tilde{l}_{ij}\right)$ of L, we have

$$L \,\mathrm{adj}\, L = (\det L)E_p = O. \tag{43}$$

Therefore $\mathrm{adj}\, L = (k_1\mathbf{x}_0 \ \ldots \ k_p\mathbf{x}_0)$, but as $\mathrm{adj}\, L$ is symmetric, we have $\mathrm{adj}\, L = k\,(\mathbf{x}_0 \ \ldots \ \mathbf{x}_0)$. $\square$

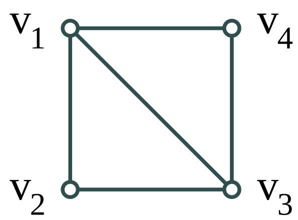


FIGURE 14

For example, let G be a graph depicted above, then the Laplacian matrix of G is

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \tag{44}$$

and $t(G)$ can be calculated by cofactors as follows.

$$|L_{11}| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8; \qquad (-1)^{3+4}|L_{34}| = -\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & -1 \end{vmatrix} = 8. \tag{45}$$

For easy graphs such as $G = C_n$, we see $t(C_n) = n$ immediately. For several special graphs G, we can derive simple formulas for $t(G)$ from Kirchhoff's theorem. Let $n = p_1 + \cdots + p_s$. We have the following.

Theorem 4.3.

$$t(K_n) = n^{n-2} \quad \textit{(Cayley's formula)}, \qquad t(K_{m,n}) = m^{n-1}n^{m-1},$$
$$t(K_{p_1,\dots,p_s}) = n^{s-2}\prod_{i=1}^{s}(n-p_i)^{p_i-1} \quad \textit{(T. L. Austin, 1960)}, \tag{46}$$
$$t(W_n) = \left(\tfrac{3+\sqrt{5}}{2}\right)^{n-1} + \left(\tfrac{3-\sqrt{5}}{2}\right)^{n-1} - 2.$$

Proof. (The first equality) By Kirchhoff's theorem,

(47)
$$t(K_n) \quad = |L_{11}| = \underbrace{\begin{vmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & -1 & n-1 \end{vmatrix}}_{n-1} = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & n-1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -1 & \dots & -1 & n-1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & n \end{vmatrix} = \underbrace{\begin{vmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & n \end{vmatrix}}_{n-2} = n^{n-2}.$$

Proof. (The second equality) The Laplacian matrix of $K_{m,n}$ is $\begin{pmatrix} nE_m & -1_{mn} \\ -1_{nm} & mE_n \end{pmatrix}$, where 1_{mn} denotes an $m \times n$ matrix consisting of only 1's. By Kirchhoff's theorem,

$$\begin{aligned} t(K_{m,n}) \quad &= \begin{vmatrix} nE_{m-1} & -1_{m-1,n} \\ -1_{n,m-1} & mE_n \end{vmatrix} = \begin{vmatrix} nE_{m-1} & -1_{m-1,n} \\ O & mE_n - \frac{m-1}{n}1_{nn} \end{vmatrix} \\ &= n^{m-1}\left|mE_n - \tfrac{m-1}{n}1_{nn}\right|. \end{aligned} \tag{48}$$

Here, as in the proof of the first equality,

(49)
$$|(p+q)E_n - q1_{nn}| = \begin{vmatrix} p & -q & \dots & -q \\ -q & p & \dots & -q \\ \dots & \dots & \dots & \dots \\ -q & -q & \dots & p \end{vmatrix} = \begin{vmatrix} p-(n-1)q & -q & \dots & -q \\ p-(n-1)q & p & \dots & -q \\ \dots & \dots & \dots & \dots \\ p-(n-1)q & -q & \dots & p \end{vmatrix}$$
$$= [p-(n-1)q]\begin{vmatrix} 1 & -q & \dots & -q \\ 1 & p & \dots & -q \\ \dots & \dots & \dots & \dots \\ 1 & -q & \dots & p \end{vmatrix} = (p+q-nq)\begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & p+q & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & p+q \end{vmatrix}$$
$$= (p+q-nq)\,(p+q)^{n-1}.$$

Hence we have

$$t(K_{m,n}) = n^{m-1}\left|mE_n - \frac{m-1}{n}1_{nn}\right| = n^{m-1}m^{n-1}. \tag{50}$$

The proofs of the third and fourth equalities are omitted.

(exercise) Calculate $t(K_{2,2,2})$ and $t(K_{3,2,1})$ by Kirchhoff's theorem.

(exercise) Prove the fourth equality of (46).

4.3 the deletion-contraction recurrence

Let $G = (V, E)$ be a multigraph. An edge contraction of $e = xy$ in G is an operation which removes e from G while simultaneously merging x and y into a new vertex z, where all edges incident to x or y are preserved to be incident to z. The multigraph obtained by this operation is written as G/e. If $e = xy$ is a multi-edge, then the edge contraction of e makes several loops zz. An edge contraction of a loop e is simply a deletion of e. Let $t_e(G)$ be the number of spanning trees of G containing e.

Theorem 4.4. *(The deletion-contraction recurrence) For a non-loop edge e, we have*

$$t(G) = t(G - e) + t_e(G) = t(G - e) + t(G/e). \tag{51}$$

Proof. The spanning trees of G are classified into ones which contain e and ones which do not contain e. Hence we have $t(G) = t(G - e) + t_e(G)$. There exists an edge contraction bijection between the spanning trees of G containing e and the spanning trees of G/e. Hence $t_e(G) = t(G/e)$. □

For a set of edges F, denote by $t_F(G)$ the number of spanning trees of G which contain all edges in F, and by G/F a multigraph obtained from G by edge contractions of all edges in F. Using Theorem 4.4 repeatedly (or classifying spanning trees by which edges of a subset E' of edges are used), we have the following.

Theorem 4.5. *(A general form of the deletion-contraction recurrence) For a subset $E' \subset E$ of edges of G, it holds that*

$$t(G) = \sum_{F \subset E'} t_{E'-F}(G - F) = \sum_{F} t\left((G - F)/(E' - F)\right), \tag{52}$$

where the summation on the right-hand side runs over all subsets F of E' such that $E' - F$ includes no cycle.

(exercise) Enumerate the numbers of spanning trees of the graphs (7),(8) in Figure 16. (hint) For the graph (8), classify the spanning trees according to which edges of e, f, g are used.

4.4 Prüfer's bijective proof of Cayley's formula

We have shown $t(K_n) = n^{n-2}$ by Kirchhoff's theorem. In this section we study a bijection from the set $T(K_n)$ of all spanning trees of K_n to the set A_n of all sequences of positive integers $\leq n$ of length $n-2$. If K_n is vertex-labeled with $1, 2, \ldots, n$, then a spanning tree of K_n is naturally interpreted into a labeled tree with the labeled vertices $1, 2, \ldots, n$.

(i) We now encode a labeled tree T on the vertices $1, 2, \ldots, n$ into a sequence belonging to A_n named the Prüfer sequence of T. Find the least leaf b_1 of T, and see the unique adjacent vertex a_1. Then remove b_1 and get a new tree T'. Find the least leaf b_2 of T', and see the unique adjacent vertex a_2. Then remove b_2 and get a new tree T''. Repeating this process, we have a sequence $(a_1, a_2, \ldots, a_{n-2})$, which is the Prüfer sequence of T.

(ii) Next we decode a Prüfer sequence $\alpha = (a_1, a_2, \dots, a_{n-2})$ into a labeled tree T. For this purpose, we restore the order of leaf deletion $\beta = (b_1, b_2, \dots, b_{n-2})$. By the algorithm (i), a labeled tree T is transformed into a tree P_2 on two vertices, removing $(n-2)$ leaves step by step. Hence we see that no leaves appear in α because a leaf is not adjacent to another leaf in a tree except for P_2, and that every non leaf vertex of degree d appears in α $(d-1)$ times. Therefore the vertices in $\{1, 2, \dots, n\}$ which do not appear in α are clearly all of the leaves of T. Thus we have the list of leaves (leaf list, for short).

Step 1: Choose the least leaf b_1 from the leaf list, which is to be removed by the algorithm (i) at the first step, and remove this leaf from the leaf list. If a_1 never appears in $(a_2, \dots, a_{n-2})$, then add a_1 as a new leaf to the leaf list.

Step 2: Choose the least leaf b_2 from the leaf list, which is to be removed by (i) at the second step, and remove it from the leaf list. If a_2 never appears in $(a_3, \dots, a_{n-2})$, then add a_2 as a new leaf to the leaf list.

.... REPEATING THIS PROCESS ....

Step $(n-2)$: Choose the least leaf b_{n-2} from the leaf list, which is to be removed by (i) at the final step, and remove it from the leaf list. Add a_{n-2} as a new leaf to the leaf list.

Now we have the order of leaf deletion $\beta = (b_1, \dots, b_{n-2})$. Together with α, a labeled tree T is reconstructed from a tree P_2 on two vertices not appearing in β, which are in the final leaf list.

These two mappings (i) and (ii) are the inverses of each other. Hence they are bijections.

Proof of Cayley's formula. By the above bijection, $t(K_n) = |T(K_n)| = |A_n| = n^{n-2}$. □

(exercise) What is the Prüfer sequence α of the labeled tree T in Figure 15? (ans): $(7, 4, 1, 1, 7, 4, 7)$. Reconstruct T from α.

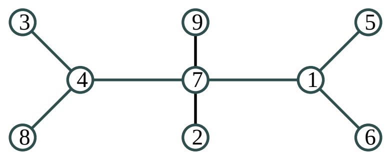


FIGURE 15

4.5 minimum spanning trees

A multigraph G is called edge-labeled if a weight (label) $w(e)$ is given to each edge e of G. Here, a weight is usually a real number. For an edge-labeled multigraph G and its submultigraph H, the weight $w(H)$ of H is the sum of all weights of the edges of H. A minimum spanning tree is a spanning tree with the minimum weight. For fixed s, a minimum spanning forest of s components is similarly defined, which is a spanning forest of s components with the minimum weight. The following are two famous algorithms to construct a minimum spanning tree of G. Suppose G has p vertices.

(Prim's algorithm) This makes one of the minimum spanning trees by developing a tree step by step from a vertex of G. Choose an arbitrary vertex $v = T_1$ of G. Take a minimum(-weighted) (non-loop) edge e_1 incident to v, and add it to v with e_1's another end vertex, then we have T_2. In general, if we have T_k on k vertices, take a minimum edge e_k connecting a vertex of T_k and a vertex not contained in T_k, and add e_k with the end vertex to T_k, then we have T_{k+1}. Repeating this process, we have a minimum spanning tree T_p.

(Proof of correctness) Let T be a minimum spanning tree of G and $T_1, T_2, \ldots, T_p$ be trees obtained by the above Prim's algorithm. If $T = T_p$, then T_p is a minimum spanning tree. Now suppose $T \neq T_p$. Let k be the least number such that T_k is not a subgraph of T. Let $e = xy$ be an edge of T_k not contained in T, where y is added to T_{k-1} with e to build T_k. Set $V = V(T_{k-1})$. Since T is a spanning tree, an (x, y)-path in T exists, and in this path there exists an edge $f = x'y'$ such that $x' \in V$ and $y' \notin V$. Then by Prim's algorithm,

$$w(e) \leq w(f). \tag{53}$$

Hence in T, we can remove f and add e to make a new minimum spanning tree T'. Repeating this process, we have $T^{(s)} = T_p$, which is a minimum spanning tree. □

(Kruskal's algorithm) We develop a spanning forest starting from the vertex set V of G into a minimum spanning tree of G. First we have a forest with no edges $V = F_0$. Next add a minimum edge e_1 to V and we have F_1. If we have F_{k-1}, add a minimum edge e_k to F_{k-1} to make F_k, such that e_k does not give any cycle in F_k. Repeating this process, we have a minimum spanning tree F_{p-1}.

(Proof of correctness) Let T be a minimum spanning tree and $F_0, F_1, \ldots, F_{p-1}$ be forests obtained by Kruskal's algorithm. If $T = F_{p-1}$, then F_{p-1} is a minimum spanning tree, thus suppose $T \neq F_{p-1}$. Let k be a least number such that F_k is not a subgraph of T, and e be an edge of F_k not contained in T, say, e is an edge added to F_{k-1} to make F_k. Here, $T \cup \{e\}$ has a cycle C. As F_k does not contain the whole of C, there exists an edge f on C which is not contained in F_k but contained in T. Since F_{k-1} is a subgraph of T, addition of either e or f keeps F_{k-1} to be a forest, and therefore, by Kruskal's algorithm, we have (53). Then removing f and adding e to T, we have a minimum spanning tree T'. Repeating this process, we have $T^{(s)} = F_{p-1}$, which is a minimum spanning tree. □

For fixed s, Kruskal's algorithm gives a minimum spanning forest of s components. In the above algorithm, F_{p-s} is simply a minimum spanning forest of s components. One can show this fact by a similar method to the above proof.

(exercise) Find minimum spanning trees of the graphs (3)–(6) in Figure 16.

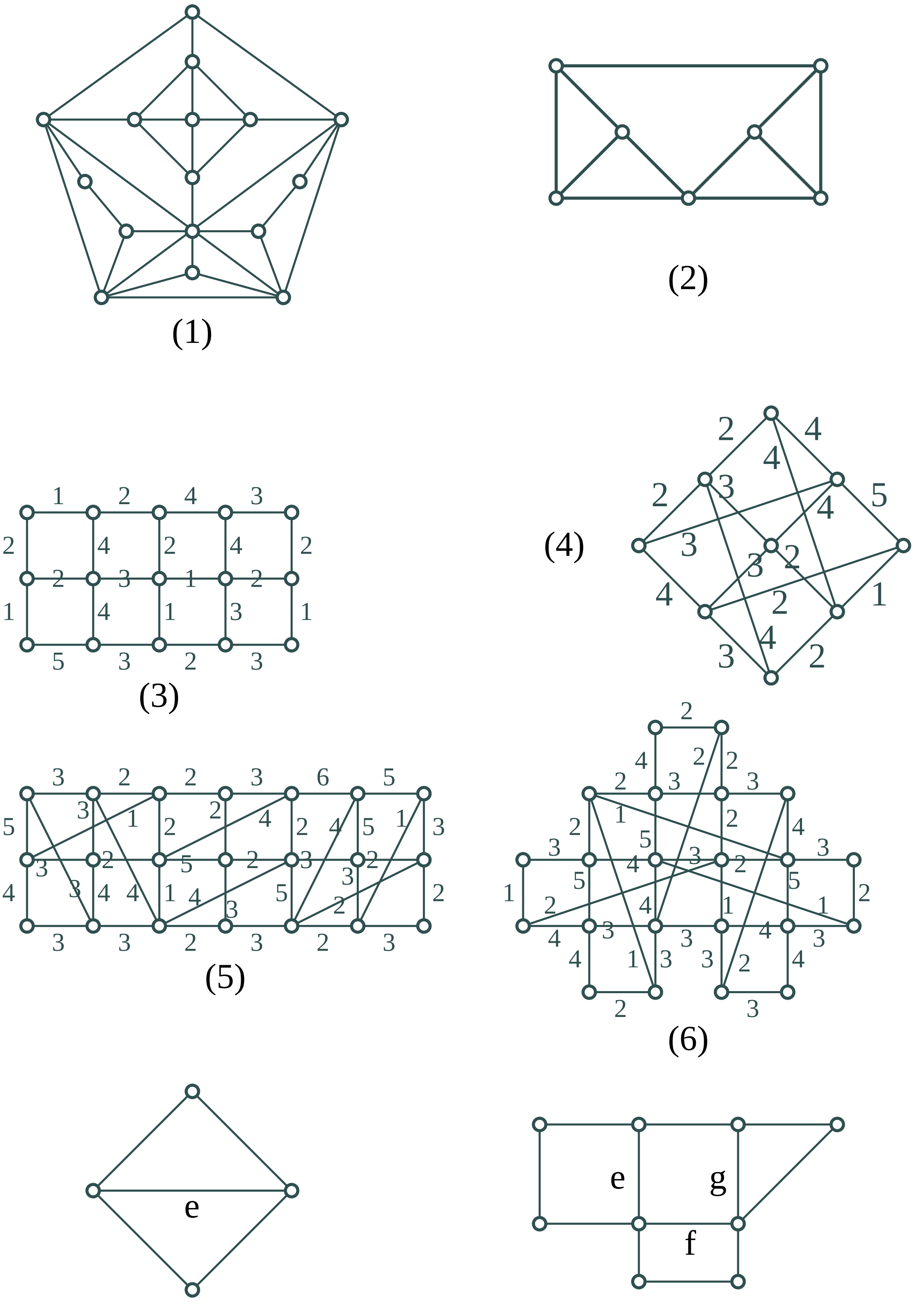


Figure 16

5 Decompositions of graphs

5.1 definition and examples

Let G be a multigraph. A set of nonempty submultigraphs $\{G_1, \ldots, G_s\}$ is called a decomposition of G, or G is said to be decomposed into $G_1, \ldots, G_s$, if the set of the edge sets of $G_1, \ldots, G_s$ is a partition of the edge set of G, say,

$$\begin{aligned} &E(G) = E(G_1) \cup \cdots \cup E(G_s) \quad \text{and} \\ &E(G_i) \cap E(G_j) = \varnothing \quad \text{for all } i \neq j. \end{aligned} \tag{54}$$

Then it is also said that $G_1, \ldots, G_s$ are packed (have a packing) into G. Usually, we suppose $E(G) \neq \varnothing$, and $E(G_i) \neq \varnothing$ for all i. In general, for multigraphs $G'_1, \ldots, G'_s$ such that $G_i \simeq G'_i$ for all i, it is said that they are packed into G. The term "pack" is also used for any supermultigraph G' of G, say, it is said that $G'_1, \ldots, G'_s$ are packed (have a packing) into G'.

For a multigraph H, an H-decomposition of G is a decomposition into multigraphs each of which is isomorphic to H. A decomposition into k-factors is called a k-factorization, and G is k-factorable if it has a k-factorization. For example, K_5 has clearly a C_5-decomposition.

Theorem 5.1. *Let G be a connected graph with an even number of edges, then G has a P_3-decomposition.*

Proof. Let G be as in the theorem. Select as many edge-disjoint copies of P_3 as possible from G. If all edges are used, a P_3-decomposition is completed, otherwise we have an even number of single edges left over in G. Since G is connected, there exists a path between any two distinct single edges. Choose two distinct single edges e_1 and e_2, and let w be a shortest path between them. The first edge of w is not a single edge, and so this forms P_3 with another edge. The location of this P_3 has three possibilities (Figure 17), and changing the location of e_1 and P_3, the resulting distance between e_1 and e_2 is shorter than before. In this way, we can get e_1 and e_2 adjacent, which make a new P_3. Repeating this process, we have a P_3-decomposition. □

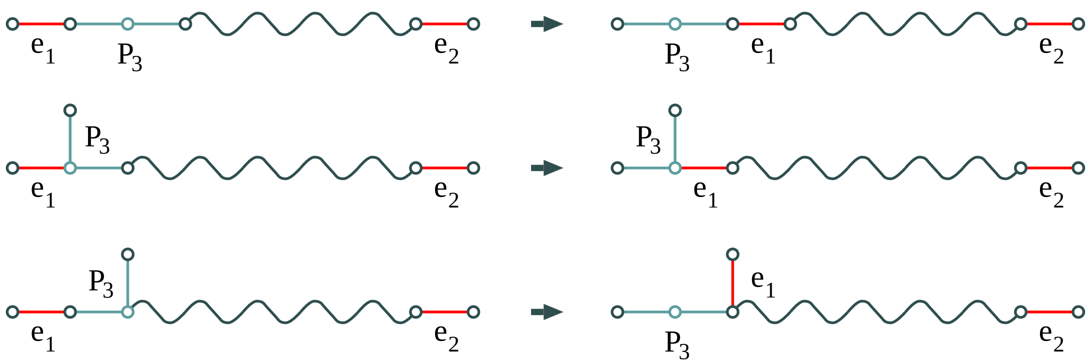


FIGURE 17

(exercise) When $m+n$ is even, find a P_3-decomposition of $L_{m,n}$.

(exercise) Is this true? A connected graph with a multiple of 3 edges is always decomposed into connected graphs with 3 edges.

5.2 decompositions of complete graphs

There are many problems concerning decompositions of graphs, especially, decompositions into fixed graphs. It is often discussed whether or not a decomposition into Hamiltonian cycles or trees is possible. One can observe easily that K_n is decomposed into Hamiltonian cycles for small odd numbers n, and this is true for all odd numbers. A tree decomposition of K_n (tree packing into K_n) is a very famous problem; a decomposition of K_n into a specified family of n trees on $1, 2, \ldots, n$ vertices, or equivalently, a packing of such a family of trees into K_n.

Theorem 5.2. *(É. Lucas, 1892) (i) The complete graph K_{2n+1} is decomposed into n Hamiltonian cycles. (ii) The complete graph K_{2n} is decomposed into $(n-1)$ Hamiltonian cycles and a 1-factor.*

Proof. (i) Consider a regular $2n$-gon with a center vertex, which locates below the $2n$-gon, say, a regular $2n$-gon pyramid, and take a Hamiltonian cycle on it as in the left graph in Figure 18. Here, note that the horizontal edge does not pass through the center vertex while the vertical edges are incident to it. Now rotate this cycle by π/n, then we have a new Hamiltonian cycle, which does not share any edges with the previous Hamiltonian cycle. Rotating the cycle $(n-1)$ times, we have n Hamiltonian cycles and have used all edges of K_{2n+1}. A Hamiltonian cycle decomposition is completed. □

FIGURE 18

(ii) By the similar method to (i). Consider a regular $(2n-1)$-gon with a center vertex. (Not necessary to avoid the same plane because no diagonal passes through the center.) A Hamiltonian cycle is shown in the right graph in Figure 18. This is composed of two 1-factors. K_{2n} is decomposed into $(2n-1)$ 1-factors, but adjacent two 1-factors form a Hamiltonian cycle. Hence K_{2n} is decomposed into $(n-1)$ Hamiltonian cycles and a 1-factor. □

The following is known as the Gyárfás tree packing conjecture:

Conjecture 5.1. *(A. Gyárfás) An arbitrary sequence of trees $T_1, T_2, \dots, T_n$, where T_i has i vertices, has a packing into K_n.*

This is an open difficult problem, however, several special cases are solved.

Theorem 5.3. *(A. Gyárfás, J. Lehel, 1976) (i) The trees $T_1, \dots, T_n$ can be packed into K_n if all but two are stars. (ii) The trees $T_1, \dots, T_n$ can be packed into K_n if there is no T_i which is different from a path or a star.*

Proof. (i) By induction on n. For $n = 1$, the proposition is true. Suppose that it is true for $n-1$.

(1) If T_n is a star S_n, then T_n and K_{n-1} are packed into K_n. Hence in this case, the proposition holds.

(2) If T_{n-1} is a star S_{n-1}, then take a leaf x of T_n, where x is incident to $e = xy$. Let $T_n - x = T'_{n-1}$. By the induction hypothesis, $T_1, \dots, T'_{n-1}$ can be packed into K_{n-1}. We can add S_{n-1} and e to K_{n-1} to make K_n reconstructing T_n by adding e to T'_{n-1} on the vertex y.

(3) Suppose neither T_n nor T_{n-1} is a star. Then T_n has a vertex x of degree $k \geq 2$ such that x is adjacent to $k-1$ leaves and a non-leaf vertex y. The closed neighborhood of x, $N = N_{T_n}[x]$ is isomorphic to S_{k+1}. Let $H = N - y$ and $T_n - H = T'_{n-k}$. By the induction hypothesis, $T_1, \dots, T'_{n-k}, \dots, T_{n-1}$ can be packed into K_{n-1}. We can add N and $T_{n-k} = S_{n-k}$ to K_{n-1} to make K_n reconstructing T_n by adding N to T'_{n-k} on the vertex y. □

The proof of (ii) is omitted.

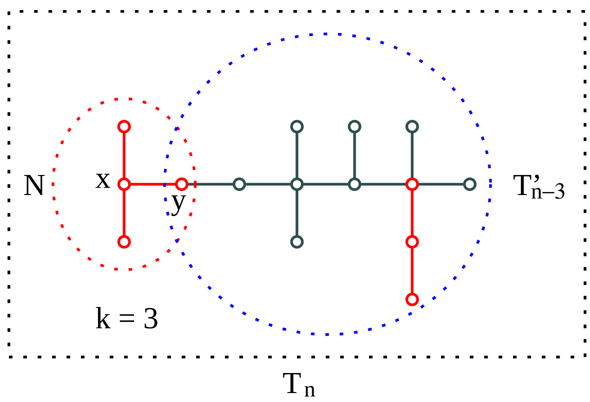


FIGURE 19

A lot of partial solutions to Conjecture 5.1 have been found. In addition, Conjecture 5.1 is generalized to the one where K_n is replaced by an n-chromatic graph. The following are the generalized conjecture and one of the recent results concerning this conjecture.

Conjecture 5.2. *(D. Gerbner, B. Keszegh, C. Palmer 2012) For* $2 \leq i \leq n$*, let* T_i *be a tree on* i *vertices. If* G *is an* n*-chromatic graph, then the set of trees* $T_2, \ldots, T_n$ *has a packing into* G*.*

Theorem 5.4. *(D. Gerbner, B. Keszegh, C. Palmer 2012) If* G *is an* n*-chromatic graph and there are at most 3 non-stars among the trees* $T_2, ..., T_n$*, then they can be packed into* G*.*

(exercise) For several n, choose trees $T_1, \ldots, T_n$ and find a packing of them into K_n.

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