POINTS AND QUESTIONS
– GRAPH THEORY –

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1 1 GRAPHS

- 1.1 -

In this section, we define several notions about graphs. We define the following:

graph, vertex set, edge set, connect (join), edgeless, independent set, end vertices, finite or infinite graph
multigraph, multi-edges, loop, multi-loops
multiset
multiplicity, order, size, finite or infinite multigraph
adjacent, incident, neighbor, independent vertices (edges)

Hereafter, we show table of new terms as above in every section. Questions concerning this section follow.

Question (Weekly report): What is the difference between graphs and multigraphs?

Question (Weekly report): Write the vertex set V and the edge set E of the center top graph in Figure 1.

- 1.2 -

We define sub(multi)graphs of a (multi)graph G, which is a graph made of several vertices and edges of G. If G' is a sub(multi)graph of G, then G is called a super(multi)graph of G'.

subgraph, submultigraph, supergraph, supermultigraph, proper
spanning subgraph, factor, induced subgraph
partial order over the set of all subgraphs
$G - V'$, $G - E'$, $G - H$
the open or closed neighborhood of a vertex

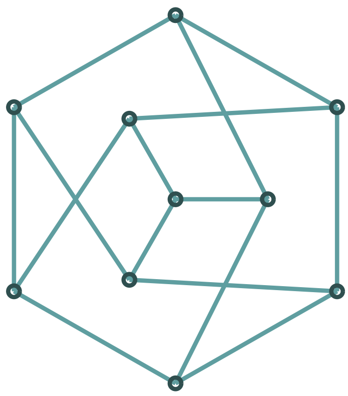
Question (Weekly report): How many spanning subgraphs of the center graph in Figure 1?

- 1.3 -

In this section, we define an important notion isomorphism of graphs. If there exists an isomorphism from G to G', then G and G' are said to be isomorphic, denoted by $G \simeq G'$. Automorphism is also defined.

isomorphism, isomorphic graphs, automorphism

Question (Final report): Find isomorphism from the right graph in the Figure 2 to the graph below. You may assign the name of vertices by yourself.



- 1.4 -

The degree of a vertex is the number of edges incident to the vertex, where loops are counted twice. Theorem 1.1 (the handshaking lemma) says that the sum of degrees of all vertices is equal to the twice of the number of edges.

even or odd vertex, isolated vertex, leaf, pendant, $\Delta(G)$, $\delta(G)$
trivial (singleton) graph, k-regular, cubic (trivalent), k-factor
degree sequence, graphic(al) sequence

Theorem 1.2 says a necessary and sufficient condition for a weakly decreasing sequence to be graphic:

A weakly decreasing sequence $d_1, d_2, \ldots, d_n$ is graphic $\iff$ $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphic.

A proof is given below of the theorem in the text. Let G be a graph with a degree sequence $d_1, d_2, \ldots, d_n$, and H be a graph with a degree sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. For the proof of $\Longleftarrow$, it suffices to construct G from H, and it is easy to add a vertex to H, and connect it to vertices of degrees $d_2 - 1, \ldots, d_{d_1+1} - 1$ to have G.

The proof of $\Longrightarrow$ is difficult, because the inverse of the above operation (adding vertex and edges) is not possible in general, the detailed method is given in the proof.

Question (Weekly report): exercises 1,3,4, page 5. For exercise 4, you can select one question from (1)–(5). Most of the questions of exercise 4 require Theorem 1.2.

Question (Final report): Find all non-isomorphic graphs with the degree sequence 2,2,2,2,2,2,2,2,2,2,2,2 (12 2's).

Question (Final report): exercise 5, page 5.

2

1 GRAPHS (continued)

- 1.5 -

In this section, we define and study walks and its families in multigraphs. A walk w in a multigraph G is an alternating sequence of vertices and edges, represented as (7), that is,

$$w = v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n,$$

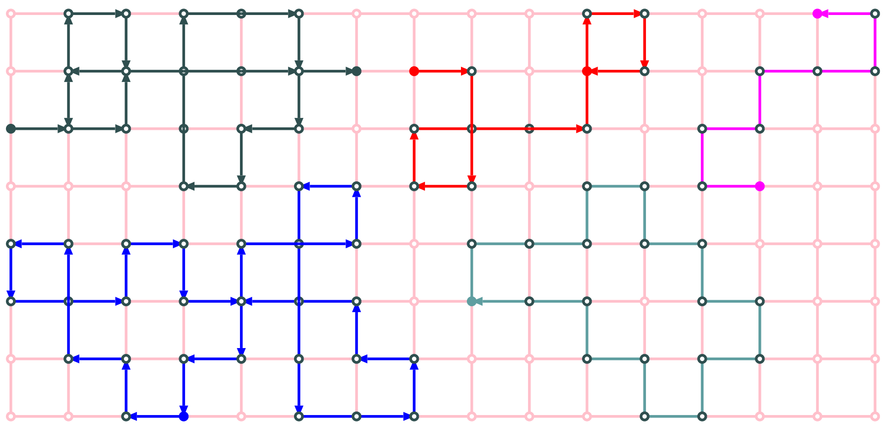
where e_i connects v_{i-1} and v_i for every $i = 1, \ldots, n$.

The length of w is defined to be n, the initial and terminal vertices (end vertices) of w are defined to be v_0 and v_n, respectively. The other vertices $v_1, v_2, \ldots, v_{n-1}$ of w are called inner vertices of w. Also, we define the terms "passes" and "meets" in the text.

If $v_0 = v_n$, w is called closed, otherwise w is called open.

In general, a walk can pass vertices and edges repeatedly, but if we add several additional conditions, we have several classes of walks: trails, paths, circuits, and cycles, which are defined by the table on page 6. The inclusion relation between them is represented by Figure 3.

The following are examples of walks, trails, paths, circuits, and cycles.



A path can be only a single vertex of length 0, a path from a vertex to itself, which is called a trivial path. The minimum length of cycles is 3 in a graph, but 1 or 2-cycles can exist in a multigraph. A walk can be represented as a sequence of vertices omitting edges if confusion does not occur. There are several additional definitions. Summary of new terms in this section follows.

walk of length n from u to v (between u and v), connected by w, initial or terminal vertex, end vertices, inner vertices, pass, w meets H, closed or open walk
trail, path, circuit, cycle, n-cycle C_n, even or odd cycle, P_n
trivial path, (u, v)-walk (trail, path) $u \longrightarrow v$
spanning (walks, circuits, subgraphs)
acyclic, unicyclic, pancyclic, girth, circumference
spanning cycle, Hamiltonian cycle, spanning path, Hamiltonian path, Hamiltonian graph, traceable, Hamiltonian connected graph
Eulerian trail, Eulerian circuit, traversable, Eulerian (multi)graph
(vertex-)independent (internally vertex-disjoint) paths, edge-independent (edge-disjoint) paths, $\kappa'(u, v)$, $\lambda'(u, v)$

Theorem 1.3 says that if a walk $u \longrightarrow v$ exists, then we have a path $u \longrightarrow v$. Also, it says that if a circuit passing an edge e exists, then we have a cycle passing e.

This is intuitively clear, because, for the former case, we have a path from a walk by taking a shortcut. The latter case is very similar.

- 1.6 -

We introduce several graphs which are very familiar in graph theory. Note that they are graphs, with no multi-edges or (multi-)loops.

complete graph K_n, complete s-partite graph $K_{p_1,\ldots,p_s}$, s-partite graph
cycle (circular) graph C_n, path graph P_n, wheel graph W_n
tree, trivial tree, n-star S_n

Let G be a graph with p vertices and q edges. Theorem 1.4 gives several conditions (ii)–(iv) equivalent to (i) G is a tree, that is, (ii) G is connected and $p = q + 1$, (iii) G is acyclic and $p = q + 1$, (iv) For any vertices u, v of G, there exists a unique path from u to v.

Question (Final report): exercises 1-3, page 9. For exercise 3, find trees on 8 vertices, only.

Hint: When you find all trees on specified number of vertices, it is convenient to classify trees by their diameters. Trees on n vertices of diameter $k-1$ can be composed by writing P_k straightly, and adding branches of $n-k$ vertices to inner vertices of P_k. Be careful to exclude isomorphic trees and not to exceed the specified diameter.

- 1.7 -

In this section we study connectivity of (multi)graphs. If a (multi)graph G has a path between every (distinct) pair of vertices, then G is called connected. A maximal connected sub(multi)graph of G is called a (connected) component of G. Hence, a (multi)graph is connected if and only if it has only one component. The left graph in Figure 4 has three components: two rectangles and one cross.

Next we define several notions for a connected multigraph G:

name	definition
cut vertex	removal of this vertex disconnects G
cut set/vertex cut/separating set	removal of this vertex set disconnects G
bridge	removal of this edge disconnects G
disconnecting set	removal of this edge set disconnects G
edge cut with respect to S, $V - S$	set of all edges connecting some vertex in S and some vertex in $V - S$

Furthermore, generalized connectivity is defined, which is fairly complicated:

k(-vertex)-connected	removal of any $k-1$ vertices keeps connected
(vertex) connectivity $\kappa(G)$	the size of minimum cut set of G or the greatest k such that G is k-connected
k-edge-connected	removal of any $k-1$ edges keeps connected
edge-connectivity $\lambda(G)$	the size of minimum disconnecting set of G or the greatest k such that G is k-edge-connected

Theorem 1.5 is a simple theorem concerning bridge, and is perhaps easier than the above definitions:

$$e \text{ is on a cycle} \iff e \text{ is not a bridge}$$

Question (Weekly report): exercises 1-2, page 10.

3

1 GRAPHS (continued)

- 1.8 -

Here we introduce distance and related topics. The distance between two vertices is defined to be the length of shortest path between them. The distance between u and v is denoted by $d(u, v)$. Then we have the axioms of distance (11).

eccentricity $e(v)$	maximum distance between v and any other vertex
diameter of $G = \text{diam}(G)$	maximum distance between any two vertices or maximum eccentricity over all vertices
radius of $G = \text{rad}(G)$	minimum eccentricity over all vertices
peripheral vertices	vertices of maximum eccentricity
center	vertices of minimum eccentricity

Strange to say, the radius is not defined to be the half of the diameter. By (12), we have

$$\text{diam}(G) \leq 2\text{rad}(G).$$

The Wiener index or polynomial of G is defined. The Wiener polynomial of G is defined by (13), say,

$$W(G; q) = \sum_{\{u,v\}} q^{d(u,v)},$$

where the sum runs over all unordered pairs of distinct vertices. Note that q is a variable. For example, if $V = \{a, b, c, d\}$, we have

$$W(G, q) = q^{d(a,b)} + q^{d(a,c)} + q^{d(a,d)} + q^{d(b,c)} + q^{d(b,d)} + q^{d(c,d)}.$$

Question (Weekly report): exercises 2-3, page 11.

Question (Final report): exercise 4, page 11.

- 1.9 -

In this section, we define graph coloring and graph labeling. For a graph G, a (vertex) coloring of G is an assignment of colors to the vertices so that adjacent vertices have distinct colors. Several relevant terms are defined, including the chromatic number $\chi(G)$ of G.

Also, an edge coloring is defined to be an assignment of colors to the edges so that adjacent edges have distinct colors. Several relevant terms are defined, including the chromatic index (edge chromatic number) $\chi'(G)$ of G.

An assignment of labels to the vertices (without any restrictions) is called a vertex labeling. Similarly, an assignment of labels to the edges is called an edge labeling.

graph coloring, (vertex) coloring, k-coloring, k-colorable, chromatic number $\chi(G)$, k-chromatic graph
color class, k-critical, critical
edge coloring, k-edge-coloring, k-edge-colorable, chromatic index (edge chromatic number) $\chi'(G)$, k-edge-chromatic graph
graph labeling, vertex labeling, edge labeling

Question (Weekly report): exercise 1, page 12. You can choose two graphs from them.

Question (Weekly report): exercise 2, page 12.

- 1.10 -

In this section, we define 3 types of matrices for multigraphs, adjacency matrix, incidence matrix, and Laplacian matrix. See the text for detailed definition. Examples of those matrices are found on page 14, exercise 1.

Theorem 1.6 is a famous fact that for the adjacency matrix A of a multigraph G, the (i, j) entry of A^n is the number of walks $v_i \longrightarrow v_j$ of length n.

Theorem 1.7 is derived from this theorem, which determine connectivity of G by matrix calculation of A.

Next, we consider the characteristic polynomial and the eigenvalues of A, the adjacency matrix of G. The adjacency matrix, like other two matrices, depends on the ordering of the vertices of G.

However, the characteristic polynomial $\Phi_A(t) = |tE - A|$ of A is uniquely determined irrespective of the order of the vertices. Hence it is determined completely by G itself, and is called the characteristic polynomial of G, denoted by $\Phi_G(t)$, and its roots are called the eigenvalues of G.

Since A is symmetric, all eigenvalues of G are real. The (muti)set of all eigenvalues of G is called the spectrum of G.

Let $G_1, \dots, G_s$ be the components of G, and the vertices and edges are ordered from G_1 to G_s. Then the adjacency, incidence and Laplacian matrices are written in a simple form as (23), where A_i, M_i, L_i are the adjacency, incidence and Laplacian matrices of Gi, respectively. From this, it follows that

$$\Phi_G(t) = \Phi_{G_1}(t)\Phi_{G_2}(t) \dots \Phi_{G_s}(t).$$

Question (Weekly report): exercise 1, page 15.

Question (Final report): exercise 2, page 15.

Question (Weekly report): exercise 3, page 15. You may calculate $\Phi_{K_3}(t)$ and $\Phi_{K_{2,2}}(t)$ instead of the original question.

Question (Final report): exercise 6, page 15.

4

2 EULERIAN/HAMILTONIAN MULTIGRAPHS

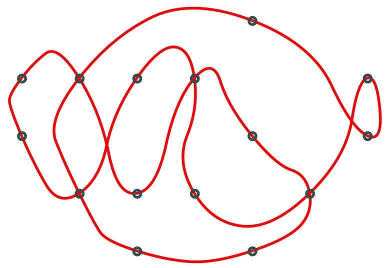
- 2.1 -

In this chapter, we study Eulerian/Hamiltonian multigraphs. The definition of them is already done in Section 1.5. Eulerian multigraphs, together with traceable multigraphs, are characterized simply by Theorem 2.1, that is,

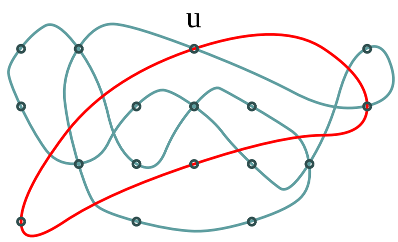
Theorem 2.1. *Let G be a finite connected (multi)graph, then*

$$\begin{aligned} G \textit{ is an Eulerian (multi)graph} \iff& G \textit{ has no odd vertices} \\ G \textit{ is traversable} \iff& G \textit{ has 0 or 2 odd vertices} \end{aligned}$$

A proof of this theorem is given in the text, but rough sketch of the proof follows. For the first equivalence (29) ($\Rightarrow$), let G be an Eulerian multigraph, and w be an Eulerian circuit. If we write w in G, which uses all edges exactly once. Then we see that all vertices are even.



For (29) ($\Leftarrow$), the proof is more difficult than ($\Rightarrow$). Let G be connected and has no odd vertices. Let w be the longest closed trail in G, and we show this is an Eulerian circuit by reduction to absurdity. Suppose w is not, then there exist edges not on w, but incident to some vertex u on w. We can make a trail w' from u using unused edges by w, then we reach to u again. Then we compose w and w' at u to get longer closed trail than w, contradiction.



A general (not necessarily connected) Eulerian multigraph is composed of a connected Eulerian component and isolated vertices. Similarly, a general traversable multigraph is composed of a connected traversable component and isolated vertices. For these graphs, we have Theorem 2.1'.

Question (Weekly report): What is an Eulerian multigraph and what is a Hamiltonian graph. (Don't copy the text, answer in your words.)

Question (Final report): Find all connected Eulerian graphs on 6 vertices (up to isomorphism).

- 2.2 -

Unlike the Eulerian case, there are no general methods to determine whether a graph is Hamiltonian or not. But concrete problems may be solved.

Question (Weekly report): Find an Eulerian circuit in the upper left graph, and a Hamiltonian cycle in the upper right graph, in Figure 7.

3 CONNECTIVITY

- 3.1 -

This chapter is devoted to connectivity of (muti)graphs. There are several important theorems, and in general, proofs of them are very difficult. We avoid to check detailed contents of the proofs, and aim to understand clearly the meaning of the theorems.

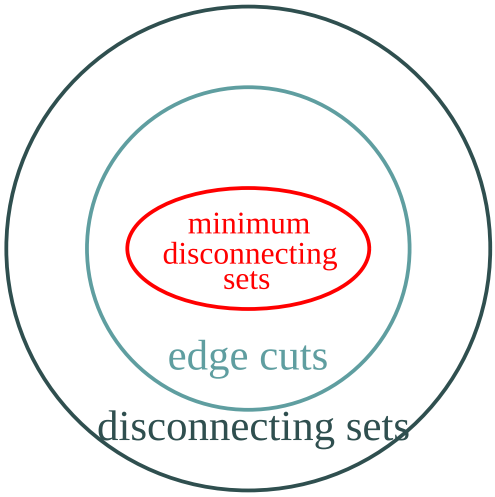
Recall the definition of vertex/edge connectivity. That is the size of a minimum cut/disconnecting set of a multigraph. But there are several specially treated graphs:

(multi)graph	vertex/edge connectivity ($\kappa(G)/\lambda(G)$)
the trivial graph	0/0
a looped vertex	0/0
disconnected (multi)graphs	0/0
K_n	$n-1/n-1$

The (vertex) connectivity of K_n is specially defined, but the edge connectivity of K_n is defined as usual.

Theorem 3.1 says that:

$$\text{a minimum disconnecting set} \Rightarrow \text{edge cut} \Rightarrow \text{disconnecting set}.$$



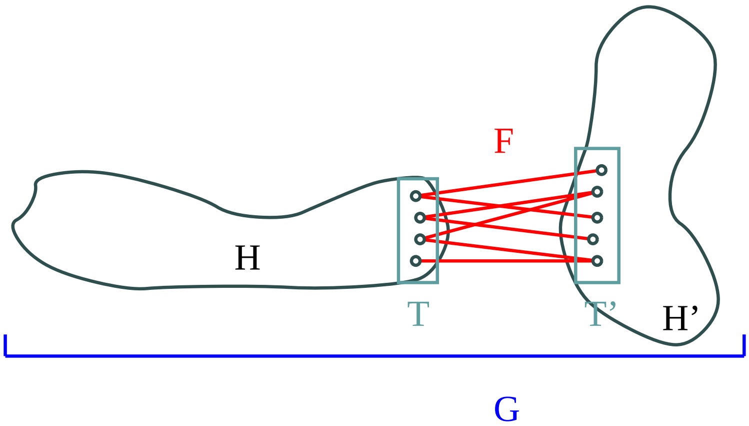
Using this theorem, we can determine the edge connectivity of K_n to be $n-1$.

Question (Final report): Determine the edge connectivity of K_n using this Theorem 3.1.

Theorem 3.2 says that for a finite multigraph G, it holds that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\kappa(G)$ is the vertex connectivity, $\lambda(G)$ is the edge connectivity and $\delta(G)$ is the smallest degree over all vertices.

Let us follow the proof. Since $\lambda(G) \leq \delta(G)$ is easy, we show $\kappa(G) \leq \lambda(G)$. We may assume G has no loops, because loops have no effect for connectivity. We also assume that G has no multi-edges, because multi-edges make only the edge connectivity increase. If $G = K_n$, then $\kappa(G) = \lambda(G)$.

Therefore we assume that G is a connected graph which is not complete. Let F be a minimum disconnecting set of s edges. By Theorem 3.1, F is an edge cut of G and $G - F$ has exactly 2 components H, H'. Let $S = V(H)$, $S' = V(H') = V - S$. Let V' be the set of end vertices of all edges in F. Let $T = S \cap V'$ and $T' = S' \cap V'$.



If $T \neq S$, then we can disconnect G by removing T to remove all of F from G, where $|T| \leq s$. Hence $\kappa(G) \leq \lambda(G)$. The case $T' \neq S'$ is treated similarly.

Consequently, we solve the case that $T = S$ and $T' = S'$. Let $|T| = m$ and $|T'| = n$. Noting that G is not complete, there are two cases: (i) there exist nonadjacent vertices $x \in T$ and $x' \in T'$, (ii) any vertices $x \in T$ and $x' \in T'$ are adjacent, but there exist vertices $x, y \in T$ or $x', y' \in T'$ such that x, y or x', y' are not adjacent.

(i): Remove vertices with edges in F step by step except x and x', then after at most s steps, F disappears and therefore G is disconnected into a graph where x and x' remain.

(ii): Let x, y be nonadjacent. Remove all vertices except x and y. Then F is removed and G is disconnected.

$$\#\text{ of removed vertices} = m + n - 2 < mn = s$$

Accordingly, it is proved that $\kappa(G) \le \lambda(G)$. □

Question (Weekly report): Prove $m + n - 2 < mn$ for all positive integers m, n.

5

3 CONNECTIVITY (continued)

- 3.2 -

In this section, we explain Menger's theorem, which is a very important one describing a relationship between the local connectivity $\kappa(u,v)$ and the maximum number $\kappa'(u,v)$ of possible independent (u,v)-paths.

Similarly, edge version of this theorem exists, which combines the local edge-connectivity $\lambda(u,v)$ and the maximum number $\lambda'(u,v)$ of possible edge-independent (u,v)-paths. Here, necessary definitions follow, note that separate u from v means that kill all (u,v)-paths.

symbol	name	definition
$\kappa(u,v)$	local connectivity	the minimum number of vertices ($\neq u,v$) to remove to separate u from v
$\lambda(u,v)$	local edge-connectivity	the the minimum number of edges to remove to separate u from v
$\kappa'(u,v)$	-	the maximum number of independent (u,v)-paths
$\lambda'(u,v)$	-	the maximum number of edge-independent (u,v)-paths
-	independent	no two of them share any vertex, except the initial and terminal ones
-	(vertex-)disjoint	no two of them share any vertex
-	edge-independent = edge-disjoint	no two of them share any edge

Then Theorem 3.3 (Menger's theorem) says that, for a finite graph G and its distinct vertices u, v,

$$\begin{aligned} &\kappa(u,v) = \kappa'(u,v) \qquad (u, v \text{ are nonadjacent}) \\ &\lambda(u,v) = \lambda'(u,v). \end{aligned}$$

We also introduce another version of Menger's theorem. To represent it, several notions are defined. Let G be a multigraph, and A, B, X be sets of vertices of G, F be a set of edges of G, H be a submultigraph of G.

name/symbol	definition
A-B path	a path from a vertex in A to a vertex in B
proper	inner vertices are not in A nor B

s-separating set X between A and B	every (proper) A-B path meets X and $\lvert X\rvert = s$
s-disconnecting set F between A and B	every (proper) A-B path meets F and $\lvert F\rvert = s$
$\kappa(G, A, B)$	the minimum size of a separating set between A and B in G
$\lambda(G, A, B)$	the minimum size of a disconnecting set between A and B in G
$\kappa^*(G, A, B)$	the maximum number of disjoint (proper) A-B paths in G
$\lambda^*(G, A, B)$	the maximum number of edge-disjoint (proper) A-B paths in G
H-path	nontrivial path which meets H exactly in its ends

It is an important point that a separating set X between A and B may contain any elements in A or B, thus it is possible that $X = A$ or $X = B$.

Then Theorem 3.4 (another version of Menger's theorem) says that, for a finite graph G and two sets A, B of vertices of G,

$$\kappa(G, A, B) = \kappa^*(G, A, B), \qquad \lambda(G, A, B) = \lambda^*(G, A, B).$$

Here, for the second equality, suppose $A \cap B = \varnothing$.

In the text, first we prove Theorem 3.4, and as a corollary, we have Theorem 3.3. The proof of Theorem 3.4 is performed by induction on the partial order over graphs, which is defined by

$$H \leq H' \iff H \text{ is a subgraph of } H'.$$

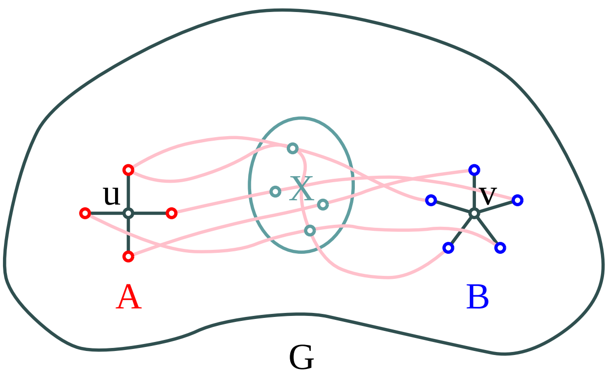
But this proof is rather difficult and we skip detailed contents.

Once we accept Theorem 3.4, the second equality of Theorem 3.3 is a special case of the second half of Theorem 3.4.

The first equality of Theorem 3.3 is derived as follows. Let A and B be the sets of all neighbors of u and v, respectively. Applying Theorem 3.4 for this A, B, we have

$$\kappa(u, v) = \kappa(G, A, B) = \kappa^*(G, A, B) = \kappa'(u, v)$$

as desired.



We can also derive Theorem 3.5, the global version of Menger's theorem, from Theorem 3.3. The theorem says that

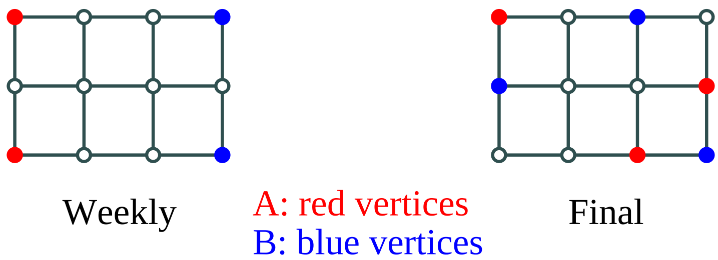
(i) G is k-connected if and only if it has k independent paths between any two distinct nonadjacent vertices,

(ii) G is k-edge-connected if and only if it has k edge-disjoint paths between any two distinct vertices.

See the text for the detailed proof.

Question (Weekly report): For the following graph on the left side, with the sets of vertices A and B, answer exercise 2 (1)–(4), page 24.

Question (Final report): Solve a similar question for the graph on the right side with A, B.



6

4 TREES

- 4.1 -