COMPLEX ANALYSIS

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1. Complex numbers and the complex plane

1.1. **Complex numbers and the complex plane.** A complex number is the number expressed as $z = x + yi$ with real numbers x and y. Here, $i = \sqrt{-1}$ represents the imaginary unit, which satisfies $i^2 = -1$. If $y = 0$, then z is a real number, hence a real number is regarded as a kind of complex number. A complex number which is not real is called an imaginary number. A number yi $(y \neq 0)$ is called a pure imaginary number. For a complex number $z = x + yi$, x is called the real part of z, denoted by $\operatorname{Re} z$, and y is called the imaginary part of z, denoted by $\operatorname{Im} z$.

For two complex numbers $z = x + yi$, $z' = x' + y'i$, it holds that

$$z = z' \iff x = x' \text{ and } y = y'. \tag{1}$$

Thus every complex number is represented as the point on the coordinate plane, whose components are the real part and the imaginary part, that is to say, $x+yi$ is represented as the point (x, y). This plane, which represents all complex numbers, is called the complex plane. The coordinate axis representing the real (respectively imaginary) parts of complex numbers is called the real (respectively imaginary) axis. The origin represents 0. The whole complex plane is identified with the set of all complex numbers $\mathbb{C}$. The real axis is identified with the set of all real numbers $\mathbb{R}$.

For $z = x + yi$, we write $\overline{z} = x - yi$, which is called the complex conjugate of z. On the complex plane, z and $\overline{z}$ are located at the symmetric positions with respect to the real axis.

1.2. **The sum and difference of complex numbers, and multiplication of complex numbers by real numbers.** A point z on the complex plane corresponds to its position vector. Thus a complex number z is represented as the vector from the origin to the point z. Of course, this vector represents the same complex number after any translation.

For complex numbers $z = x + yi$, $z' = x' + y'i$, define the sum $z + z'$ of them by

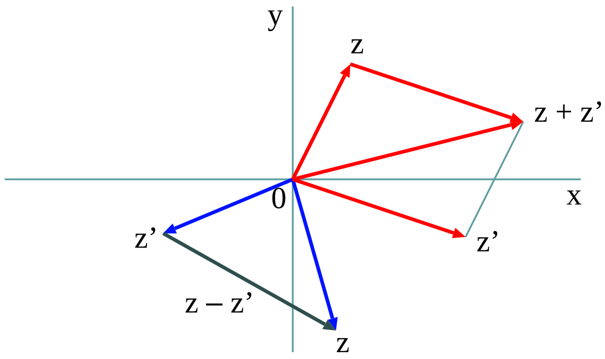
$$z + z' = (x + x') + (y + y')i. \tag{2}$$

On the complex plane, we have, as vectors,

$$(x, y) + (x', y') = (x + x', y + y'), \tag{3}$$

say, the sum of complex numbers is represented as the sum of their position vectors.

Key words and phrases. complex numbers, complex plane, polar form, de Moivre's theorem, $z^n = c$, Riemann sphere.



Obviously, we have $z + z' = z' + z,\ (z + z') + z'' = z + (z' + z'')$.

For a complex number z, $-z$ is defined as the complex number satisfying that $z + (-z) = (-z) + z = 0$, which is the inverse vector of the vector z as a vector, or which is the point symmetric to z with respect to the origin as a point, say, $-z' = -(x' + y'i) = -x' - y'i$. Hence the difference $z - z' = z + (-z')$ between z and z' is expressed as

$$z - z' = (x - x') + (y - y')i. \tag{4}$$

On the complex plane, we have

$$(x, y) - (x', y') = (x - x', y - y'), \tag{5}$$

which is represented as the vector in the above figure. This figure is obviously valid also because $z' + (z - z') = z$.

A real multiple of a complex number z is defined by

$$x'z = x'x + x'yi. \tag{6}$$

This operation is written as $x'(x, y) = (x'x, x'y)$ on the complex plane, which corresponds to scalar multiplication of a vector. For positive x', $x'z$ is a vector obtained by multiplying the magnitude by x' without changing the direction, whereas $-x'z$ is a vector obtained by multiplying the magnitude by x' but inverting the direction.

1.3. **The product, quotient of complex numbers and polar form.** We define the product of complex numbers assuming that distributive law holds:

$$zz' = (x + yi)(x' + y'i) = (xx' - yy') + (xy' + x'y)i. \tag{7}$$

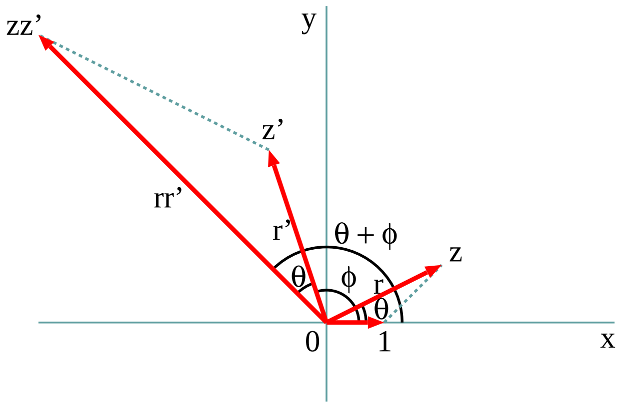
The following laws hold: $zz' = z'z,\ (zz')z'' = z(z'z''),\ z(z' + z'') = zz' + zz'',\ (z + z')z'' = zz'' + z'z''$.

Here, we introduce polar form to understand the product of complex numbers on the complex plane. For a complex number $z = x + yi$, the distance r between the origin and the point z on the complex plane is called the absolute value (or modulus or magnitude) of z, denoted by $|z|$. This is the length of the vector representing z. By the Pythagorean theorem, we have $|z| = \sqrt{x^2 + y^2}$. Also, the general angle θ from the positive real axis to the vector representing z is called the argument of z, denoted by $\arg z$. Then we see that

$$z = r(\cos\theta + i\sin\theta). \tag{8}$$

This expression is called the polar form of z. In polar form, z is indicated by the absolute value and the argument of z. Let us calculate the product of complex numbers in polar form.

$$\begin{aligned} zz' &= r(\cos\theta + i\sin\theta)\, r'(\cos\varphi + i\sin\varphi) \\ &= rr'\left[(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)\right] \\ &= rr'\left[\cos(\theta+\varphi) + i\sin(\theta+\varphi)\right]. \end{aligned} \tag{9}$$



From this result, it follows that the absolute value of the product of complex numbers is equal to the product of the absolute values of them, and the argument of the product is equal to the sum of the arguments:

$$|zz'| = |z||z'|, \qquad \arg(zz') = \arg z + \arg z'. \tag{10}$$

For the product of more than 2 complex numbers, similar formula holds. Here, consider z^n for any positive integer n, then we have the following de Moivre's theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta). \tag{11}$$

For the quotient of complex numbers, by making the denominator real, we see the result is also a complex number. Indeed,

$$\tfrac{x'+y'i}{x+yi} = \tfrac{(x'+y'i)(x-yi)}{(x+yi)(x-yi)} = \tfrac{xx'+yy'+(xy'-x'y)i}{x^2+y^2} = \tfrac{xx'+yy'}{x^2+y^2} + \tfrac{xy'-x'y}{x^2+y^2}i. \tag{12}$$

Also, in (9), letting $rr' = s$, $\theta + \varphi = \rho$, we have

$$\begin{gathered} \tfrac{s(\cos\rho + i\sin\rho)}{r(\cos\theta + i\sin\theta)} = \tfrac{s}{r}\left[\cos(\rho-\theta) + i\sin(\rho-\theta)\right], \\ |z'/z| = |z'|/|z|, \quad \arg(z'/z) = \arg z' - \arg z. \end{gathered} \tag{13}$$

In particular, putting $z' = 1$, we have

$$|1/z| = 1/|z|, \qquad \arg(1/z) = -\arg z. \tag{14}$$

From this and (11), it follows that

$$z^{-n} = 1/z^n = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)]. \tag{15}$$

That is, de Moivre's theorem is valid also for negative integer powers.

In the polar form (8), we often write $\cos\theta + i\sin\theta$ simply as $e^{i\theta}$, say, define[1]

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{16}$$

[1] (16) is called Euler's formula.

Then the polar form of z is written as

$$z = re^{i\theta}. \tag{17}$$

Especially, $e^{i\theta}$ is a complex number with absolute value 1 and argument θ, thus it lies on the unit circle. By using this notation, de Moivre's theorem is expressed as follows. For every integer n,

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}. \tag{18}$$

(exercise01) Calculate the following: (1) $(1+\sqrt{3}i)^{14}$. $(2^{14}e^{\frac{2\pi}{3}i})$ (2) $(-1-i)^{10}$. $(32i)$
(exercise02) Show the following: (1) $z\overline{z} = |z|^2 = x^2 + y^2$. (2) $\overline{z \pm z'} = \overline{z} \pm \overline{z'}$. (3) $\overline{zz'} = \overline{z}\,\overline{z'}$. (4) $\overline{z/z'} = \overline{z}/\overline{z'}$. (5) $||z| - |z'|| \leq |z \pm z'| \leq |z| + |z'|$.

1.4. **The equation $z^n = c$.** Let c be a fixed complex number (complex constant). We often need to find all complex numbers satisfying the equation $z^n = c$, say, the solution to $z^n = c$. Let us solve this by using de Moivre's theorem. Letting $z = re^{i\theta}$, and $c = se^{i\varphi}$.

$$z^n = c \iff r^n e^{in\theta} = se^{i\varphi} \tag{19}$$

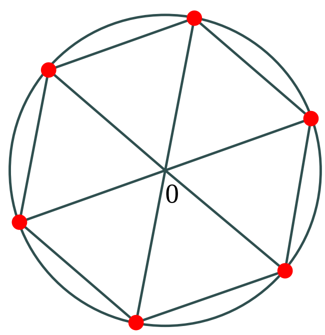
Comparing the absolute values and the arguments of both sides of the right equation, we have

$$\begin{array}{ll} r^n = s. & \therefore\ r = \sqrt[n]{s}. \\ n\theta = \varphi + 2m\pi \quad (m \in \mathbb{Z})^2. & \therefore\ \theta = \frac{\varphi + 2m\pi}{n}. \end{array} \tag{20}$$

Consequently, we have the solution to $z^n = c$:

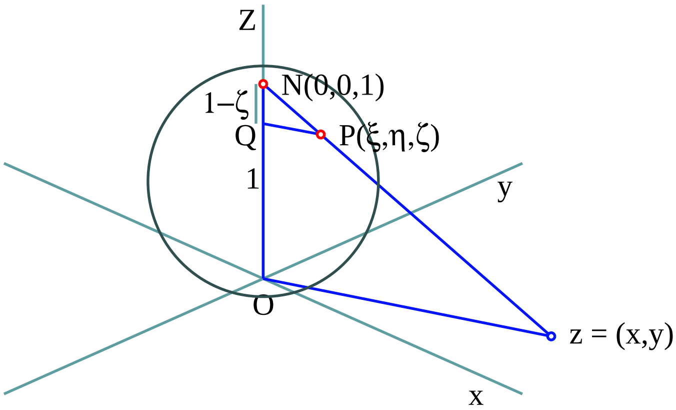
$$z = \sqrt[n]{s}e^{i\frac{\varphi+2m\pi}{n}}. \tag{21}$$

Here, m runs over all integers, but actually, after running $m = 0, 1, \ldots, n-1$, $m = n$ turns the value of z to the one with $m = 0$, and then repeats this process. Hence we consider only the cases $m = 0, 1, \ldots, n-1$. These values of z have the same absolute values, and as m increases by 1, the argument increases by $\frac{2\pi}{n}$. This means that all solutions z form the vertices of a regular n-polygon.
(exercise03) Solve $z^5 = 32$. $[z = 2e^{\frac{2m}{5}\pi i}\ (m = 0, 1, 2, 3, 4)]$



[2] $\mathbb{Z}$ represents the set of all integers.

1.5. **Expression of sets.** Sometimes we want to express sets on the complex plane by equations or inequalities. Here we give simple examples. Let r be a positive constant, and consider the equation $|z| = r$. This means that the distance between z and the origin is equal to r, hence z lies everywhere on the circle with the center at the origin and radius r. Therefore the equation represents the circle. Similarly, $|z - c| = r$ means the distance between z and c is equal to r, and represents the circle with the center at c and radius r. Furthermore, $|z - c| \leq r$ represents the circle and the inner area of it. For more complicated equations, letting, for example, $z = x + iy$ and derive the relations between x and y.
(exercise04) Illustrate the area of z satisfying $z^2 + \overline{z^2} \leq 4$.



1.6. **The Riemann sphere.** A sphere K of diameter 1 is on the complex plane, touching at the origin. Every complex number is represented as a point on K as follows. Set up Z-axis vertically to the complex plane, intersecting with K at the origin and $N(0, 0, 1)$. For a point (x, y) on the complex plane, the intersecting point of the line connecting N and (x, y), and K is uniquely determined, and denote it by $P(\xi, \eta, \zeta)$. If (x, y) moves then P also moves, and for any $P \neq N$, the corresponding point (x, y) exists, thus the complex plane corresponds one-to-one to $K - N$. This correspondence is called the stereographic projection, which represent a complex number $x + yi$ as the corresponding point $P(\xi, \eta, \zeta)$ on K.

If P comes closer to N, then the corresponding point (x, y) go further from the origin, and so we identify N with the point of infinity ∞. Consequently, K is regarded as representing $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$, and called the Riemann sphere or the complex sphere.

If we compare the Riemann sphere to Earth, correspondence examples of the stereographic projection follows.

the complex plane and ∞	the Riemann sphere
∞	the North Pole
$\lvert z\rvert > 1$	the Northern Hemisphere
a unit circle	the Equator
$\lvert z\rvert < 1$	the Southern Hemisphere
0	the South Pole

Here, we study the formula between x, y and ξ, η, ζ. Since P is on the sphere K,

$$\xi^2 + \eta^2 + \zeta^2 = \zeta. \tag{22}$$

As $\triangle ONz \sim \triangle QNP$,

$$1 : 1 - \zeta = x : \xi = y : \eta. \tag{23}$$

$$\therefore \quad \xi = (1-\zeta)x, \quad \eta = (1-\zeta)y. \tag{24}$$

$$\therefore \quad x = \tfrac{\xi}{1-\zeta}, \quad y = \tfrac{\eta}{1-\zeta}. \tag{25}$$

Next, substituting (24) into (22), we have

$$\begin{gathered} (1-\zeta)^2x^2 + (1-\zeta)^2y^2 + \zeta^2 = \zeta. \\ \therefore \quad (1-\zeta)(x^2+y^2) = \zeta. \qquad \therefore \quad \zeta = \tfrac{x^2+y^2}{1+x^2+y^2}. \\ \text{By (24)}, \quad \xi = \tfrac{x}{1+x^2+y^2}, \quad \eta = \tfrac{y}{1+x^2+y^2}. \end{gathered} \tag{26}$$

(note) Every line on the complex plane corresponds to some circle on the Riemann sphere passing ∞, and every circle on the complex plane corresponds to some circle on the Riemann sphere not passing ∞. This fact is called a circle-to-circle correspondence, which we prove below. Let a circle on the complex plane be $x^2 + y^2 + ax + by + c = 0$. Then by (25), the corresponding $P(\xi, \eta, \zeta)$ satisfies that

$$\begin{gathered} \left(\tfrac{\xi}{1-\zeta}\right)^2 + \left(\tfrac{\eta}{1-\zeta}\right)^2 + a\tfrac{\xi}{1-\zeta} + b\tfrac{\eta}{1-\zeta} + c = 0. \\ \text{Hence by (22)}, \ \zeta + a\xi + b\eta + c(1-\zeta) = 0. \end{gathered} \tag{27}$$

This shows that P is on some plane, and P is also in the sphere, therefore on some circle. It is clear that this circle does not pass ∞. A proof for lines on the complex plane is similar.

CHAPTER 2

CONDITIONS FOR COMPLEX DIFFERENTIABILITY

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KEYWORDS: COMPLEX FUNCTIONS, REAL FUNCTIONS OF TWO VARIABLES, PARTIAL DERIVATIVES, TOTALLY DIFFERENTIABLE, DERIVATIVES, CAUCHY–RIEMANN EQUATIONS, COMPLEX DOMAINS, HOLOMORPHIC FUNCTIONS, HARMONIC FUNCTIONS

2.1. **Complex functions.** A complex function is a function from some set of complex numbers to another set of complex numbers. A complex function f which maps z to w is denoted by $w = f(z)$. Here, since z moves some set of complex numbers, z is called a complex independent variable, and since w moves dependent on z, w is called a complex dependent variable. For simplicity, we usually call z and w complex variables or variables. A complex function $f(z)$ is expressed using the real part and the imaginary part as $f(z) = u(z) + iv(z)$, where $z = x + iy$ and z corresponds one-to-one to the pair (x, y). Hence we have

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y). \tag{1}$$

We sometimes write simply as $f(z) = u+iv$. The complex plane representing the value of z is called the z-plane, and the complex plane representing the value of $w = f(z)$ is called the w-plane. The set D where the variable z of a complex function $f(z)$ moves is called the domain of $f(z)$. The set of all values of $f(z)$, that is, $\{f(z) \mid z \in D\}$ is called the image of $f(z)$ or the image of D by $f(z)$, denoted by $f(D)$.

Sometimes we write a complex function, not using variables, but as

$$f : D \longrightarrow E. \tag{2}$$

This is a complex function f which maps every point of D to some point of E; D is the domain of f, and E is the codomain of f. In general, $f(D) \subset E$, however, if $f(D) = E$, then f is called a surjection. If $z \neq z' \implies f(z) \neq f(z')$, then f is called an injection. If f is both a surjection and an injection, it is called a bijection or one-to-one correspondence.

(exercise01) If $f(z) = z^2$, determine u and v.

(exercise02) Determine the image of given D by also given $f(z)$. Illustrate on the w-plane. (The equation of the image follows.) (1) $f(z) = iz$, $D : |z - 2i| \leq 1$. $(|w + 2| \leq 1)$ (2) $f(z) = z^2$, $D : 2 \leq |z| \leq 3$. $(4 \leq |w| \leq 9)$ (3) $f(z) = z^2 - 6z$, $D : |z - 3| \leq 2$. $(|w + 9| \leq 4)$

2.2. **Real functions of two variables.** As we saw in (1), every complex function is composed of two real functions of two variables. So we introduce some properties of real functions of two variables (two-variable functions or functions, for simplicity). Continuity and differentiability of two-variable functions is rather complicated than ones of one-variable functions. First of all, to visualize a two-variable function $u(x,y)$, we prepare its graph as a surface in the xyz-space. Intuitively speaking, $u(x,y)$ is called continuous if its graph is not torn, but more precisely, $u(x,y)$ is called continuous at a point (x_0,y_0) if the following is satisfied.

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u(x_0,y_0) \tag{3}$$

Here, there are infinitely many direction from which (x,y) gets close to (x_0,y_0). This equation means that $(x,y)\to(x_0,y_0)$ from any direction implies $u(x,y)\to u(x_0,y_0)$. If $u(x,y)$ is continuous at every point in some set D under consideration, then $u(x,y)$ is called continuous in D or simply continuous.

A function $u(x,y)$ is partially differentiable at a point (x,y) if the following limits exist.

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) \equiv u_x(x,y) &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}\\ \frac{\partial u}{\partial y}(x,y) \equiv u_y(x,y) &= \lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}\end{aligned} \tag{4}$$

They are called the partial differential coefficient of $u(x,y)$ at (x,y) with respect to x and y, respectively from top to bottom. If $u(x,y)$ is partially differentiable at every point in some set D under consideration, then $u(x,y)$ is partially differentiable in D or simply partially differentiable. Then both of (4) are regarded as new functions of x and y, and so they are called the partial derivatives of $u(x,y)$. In more detail, $u_x(x,y)$ (respectively, $u_y(x,y)$) is the partial derivative of $u(x,y)$ with respect to x (respectively, y). The operation to derive $u_x(x,y)$ (respectively, $u_y(x,y)$) from $u(x,y)$ is called the partial differentiation of $u(x,y)$ with respect to x (respectively, y). If both of the partial derivatives (4) of $u(x,y)$ are partially differentiable (in D), then $u(x,y)$ is called twice partially differentiable (in D). All functions derived by partially differentiating $u(x,y)$ twice are called the second partial derivatives of $u(x,y)$. They are denoted by

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(x,y) = u_{xx}(x,y)\\ \frac{\partial^2 u}{\partial y\partial x}(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = u_{xy}(x,y)\\ \frac{\partial^2 u}{\partial x\partial y}(x,y) &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(x,y) = u_{yx}(x,y)\\ \frac{\partial^2 u}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)(x,y) = u_{yy}(x,y)\end{aligned} \tag{5}$$

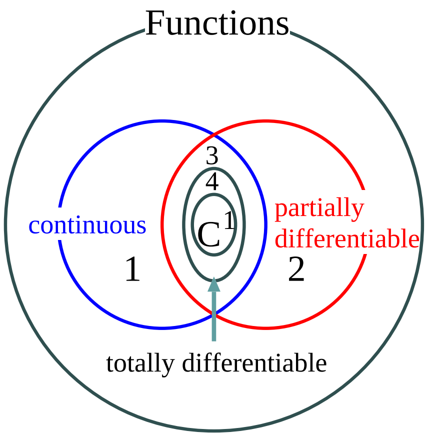
Similarly, in general, we can define the notions of r times partially differentiable, and the r-th partial derivatives (partial derivatives of order r). If a function is continuous, r times partially differentiable, and all partial derivatives of order at most r are continuous, then the function is said to be of class C^r or a C^r-function.[1] A C^0-function is a continuous function. If a function is of class C^r for all positive integer r, it is said to be of class C^∞. Polynomials, exponential functions, the trigonometric functions sin, cos, and their compositions, etc. are of class C^∞ (in the whole complex plane). A function $u(x,y)$ of class C^2 satisfies that $u_{xy} = u_{yx}$. This means that the order of partial differentiations is interchangeable. In general, we have

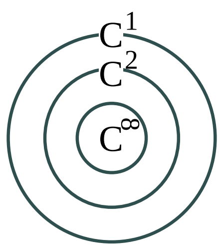
Theorem 1. *For every function of class C^r, the order of partial differentiations at most r times is interchangeable.*

There is another notion "(totally) differentiable" which we will use later. This is defined as, in the first equation of (14), $\Delta x, \Delta y \to 0$ implies $\frac{\epsilon_1}{\sqrt{\Delta x^2+\Delta y^2}} \to 0$. More intuitively speaking, the graph of a function is approximated by a plane in a sufficiently small region.

The notions appearing in this section concerning continuity and differentiation are defined similarly for n-variable functions $u(x_1, \dots, x_n)$.[2] Now what is the relationship between them? It is represented by a Venn diagram as follows. Proofs of these facts are omitted. In addition, we give sample functions which belong to the regions numbered 1 to 4, respectively. Here $z = x + iy$, and for the samples 2–4, the value at the origin id defined to be 0.

1: $|z|$. 2: $\frac{xy}{x^2+y^2}$. 3: $\mathrm{sgn}(xy)\sqrt{|xy|}\sin\frac{\sqrt{|xy|}}{x^2+y^2}$. 4: $|z|^2\cos\frac{1}{|z|}$.





[1] A function of class C^1 is also called continuously differentiable, and a function of class C^r is also called r times continuously differentiable.

[2] Theorem 1 holds for any n-variable functions.

2.3. **Complex derivatives.** A complex function $f(z)$ is defined to be continuous at a point c if the following holds.

$$\lim_{z \to c} f(z) = f(c) \tag{6}$$

Also, $f(z)$ is called continuous in a set D if $f(z)$ is continuous at every point in D. The equation (6) has a strong meaning that $z \to c$ in any path implies the convergence of $f(z)$ to $f(c)$. Here, letting $z = x + yi$, $c = a + bi$, (6) is rewritten as

$$\begin{aligned} &\lim_{(x,y)\to(a,b)} [u(x,y) + iv(x,y)] = u(a,b) + iv(a,b) \\ \iff &\begin{cases} \displaystyle\lim_{(x,y)\to(a,b)} u(x,y) = u(a,b) \\ \displaystyle\lim_{(x,y)\to(a,b)} v(x,y) = v(a,b). \end{cases} \end{aligned} \tag{7}$$

Hence, $f(z)$ is continuous at a point c if and only if u and v are continuous at a point (a,b), and therefore, $f(z)$ is continuous in a set D if and only if u and v are continuous in D.

A complex function $f(z)$ defined to be differentiable at a point z if there exists a limit:

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z). \tag{8}$$

This is called the differential coefficient of $f(z)$ at z. Further, $f(z)$ is differentiable in a set D if $f(z)$ is differentiable at every point in D. Then we have a complex function $f'(z)$ which maps every point z in D to $f'(z)$. This $f'(z)$ is called the derivative of $f(z)$. The operation to derive $f'(z)$ from $f(z)$ is called the differentiation of $f(z)$. If $f'(z)$ is again differentiable in D, then $f(z)$ is called twice differentiable and the derivative of $f'(z)$ is denoted by $f''(z) = f^{(2)}(z)$, which is called the second derivative of $f(z)$. Similarly, we define the notions of n times differentiable and the n-th derivative (derivative of order n).[3]

The equation (8) has, similarly to (6), a strong meaning that $\Delta z \to 0$ in any path implies the convergence of the limit to exactly one limit value. This condition is stronger than the case of ordinary real differentiation even if the definition is given by the same formula.

2.4. **Cauchy–Riemann equations.** Now suppose that a complex function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point z, say, the limit (8) exists. Letting $\Delta z = \Delta x + i\Delta y$, the contents of the limit (8) is represented by u and v as follows.

$$\begin{aligned} &\tfrac{1}{\Delta x + i\Delta y}[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)] \\ = &\tfrac{1}{\Delta x + i\Delta y}[u(x+\Delta x, y+\Delta y) - u(x,y)] + \tfrac{i}{\Delta x + i\Delta y}[v(x+\Delta x, y+\Delta y) - v(x,y)] \end{aligned} \tag{9}$$

Since $f(z)$ is differentiable by hypothesis, the limit (9) has one value $f'(z)$, whatever the direction of $\Delta z \to 0$. First, we differentiate along x-axis. Then we put $\Delta y = 0$ in

[3]For a function $w = f(z)$, we sometimes write as $f'(z) = \frac{dw}{dz}$, $f''(z) = \frac{d^2w}{dz^2}, \ldots, f^{(n)}(z) = \frac{d^nw}{dz^n}$.

(9).

$$\begin{aligned}\lim_{\Delta x\to 0}&\left[\tfrac{1}{\Delta x}[u(x+\Delta x,y)-u(x,y)]+\tfrac{i}{\Delta x}[v(x+\Delta x,y)-v(x,y)]\right]\\&=u_x(x,y)+iv_x(x,y)=f'(z).\end{aligned}\tag{10}$$

Similarly, differentiate along y-axis. Letting $\Delta x=0$ on the right-hand side of (9),

$$\begin{aligned}\lim_{\Delta y\to 0}&\left[\tfrac{1}{i\Delta y}[u(x,y+\Delta y)-u(x,y)]+\tfrac{i}{i\Delta y}[v(x,y+\Delta y)-v(x,y)]\right]\\&=\tfrac{1}{i}u_y(x,y)+v_y(x,y)=f'(z).\end{aligned}\tag{11}$$

Hence we have

$$u_x(x,y)+iv_x(x,y)=v_y(x,y)-iu_y(x,y)=f'(z),\tag{12}$$

and therefore,

$$\begin{cases}u_x(x,y)=v_y(x,y)\\u_y(x,y)=-v_x(x,y).\end{cases}\tag{13}$$

This equations are called the Cauchy–Riemann equations, which give a necessary condition for the differentiability of $f(z)$.

2.5. **Conditions for differentiability.** Next we study the necessary and sufficient conditions for the differentiability of $f(z)$ at a point z. For differentiability, (13) is necessary, thus we proceed with this presupposition hereafter. First, let ϵ_1 and ϵ_2 be errors of linear approximations of u and v by using differentiation, respectively.

$$\begin{aligned}u(x+\Delta x,y+\Delta y)-u(x,y)&=u_x(x,y)\Delta x+u_y(x,y)\Delta y+\epsilon_1\\v(x+\Delta x,y+\Delta y)-v(x,y)&=v_x(x,y)\Delta x+v_y(x,y)\Delta y+\epsilon_2\end{aligned}\tag{14}$$

From these, it follows that

$$\begin{aligned}f(z+\Delta z)-f(z)&=u(x+\Delta x,y+\Delta y)-u(x,y)+i[v(x+\Delta x,y+\Delta y)-v(x,y)]\\&=u_x(x,y)\Delta x+u_y(x,y)\Delta y+\epsilon_1+i[v_x(x,y)\Delta x+v_y(x,y)\Delta y+\epsilon_2].\end{aligned}$$

By the Cauchy–Riemann equations,

$$\begin{aligned}&=u_x(x,y)\Delta x-v_x(x,y)\Delta y+\epsilon_1+i[v_x(x,y)\Delta x+u_x(x,y)\Delta y+\epsilon_2]\\&=[u_x(x,y)+iv_x(x,y)](\Delta x+i\Delta y)+\epsilon_1+i\epsilon_2.\end{aligned}$$

Letting $\epsilon_1+i\epsilon_2=\epsilon$, and noting that $\Delta z=\Delta x+i\Delta y$,

$$=[u_x(x,y)+iv_x(x,y)]\Delta z+\epsilon.\tag{15}$$

Consequently, we have

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=u_x(x,y)+iv_x(x,y)+\frac{\epsilon}{\Delta z}\tag{16}$$

Here, let us assume that $f(z)$ is differentiable at a point z. As we have seen, then (12) holds, hence (16) is rewritten as

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=f'(z)+\frac{\epsilon}{\Delta z},\tag{17}$$

and therefore it is necessary for the left-hand side to converge to $f'(z)$ that

$$\frac{\epsilon}{\Delta z}\to 0.\tag{18}$$

Conversely, if we assume (18), the left-hand side of (16) converges to $u_x(x,y) + iv_x(x,y)$, which shows that $f(z)$ is differentiable at a point z.

In this way, we see that (18) is the necessary and sufficient condition for $f(z)$ to be differentiable at a point z. We have supposed, however, that the necessary condition (13) as a major premise, thus the necessary and sufficient conditions are both of (13) and (18).

Here it holds that

$$\frac{\epsilon}{\Delta z} = \frac{\epsilon_1}{\Delta z} + i\frac{\epsilon_2}{\Delta z}, \tag{19}$$

and hence

$$\frac{\epsilon}{\Delta z} \to 0 \iff \frac{\epsilon_1}{\Delta z} \to 0 \quad \text{and} \quad \frac{\epsilon_2}{\Delta z} \to 0, \tag{20}$$

or equivalently,

$$\frac{\epsilon_1}{\sqrt{\Delta x^2 + \Delta y^2}} \to 0 \quad \text{and} \quad \frac{\epsilon_2}{\sqrt{\Delta x^2 + \Delta y^2}} \to 0. \tag{21}$$

This means that u and v are totally differentiable at a point (x,y). By the use of this term, we have the following.

Theorem 2. *The necessary and sufficient condition for $f(z) = u(x,y) + iv(x,y)$ to be differentiable at a point $z = x + yi$ is that u and v are totally differentiable and satisfy the Cauchy–Riemann equations at a point (x,y).*

2.6. **Fundamental terminologies and conditions for holomorphy.** Hereafter, we introduce fundamental terminologies concerning topology and differential calculus. An open disc is the interior (excluding the boundary) of a circle of positive radius. Open discs of center z are called neighborhoods of z. There are infinitely many neighborhoods of z.

Let S be a set on the complex plane. If no neighborhood of a point z is contained in S or the complement S^c of S, then z is called a boundary point of S. The set of all boundary points of S is called the boundary of S, denoted by ∂S. Define the interior S^o, closure $\overline{S}$, exterior S^e of S as follows.

$$\begin{aligned} &S^o = S - \partial S \qquad \overline{S} = S \cup \partial S \\ &S^e = (S^c)^o = \overline{S}^c \end{aligned} \tag{22}$$

The whole complex plane $\mathbb{C}$ is divided into

$$\mathbb{C} = S^o \cup \partial S \cup S^e. \tag{23}$$

A point in S^o is an interior point of S, a point in S^e is an exterior point of S. This is equivalent to the definition that if some neighborhood of z is contained in S, then z is an interior point; and if some neighborhood of z is contained in S^c, then z is an exterior point. It holds that

$$z \text{ is an exterior point of } S \iff z \text{ is an interior point of } S^c. \tag{24}$$

A set which satisfies that $S = S^o$ is called an open set, whereas a set which satisfies that $S = \overline{S}$ is called a closed set. In other words, a set with no boundary points is an open set, whereas a set with all boundary points is a closed set. We have

$$S \text{ is an open set} \iff S^c \text{ is a closed set.} \tag{25}$$

The whole complex plane and open discs are examples of open sets. An open set allows several holes if it contains no boundary point of the holes. Also, an open set can be disconnected into several components. A closed set also has holes and can be disconnected. A closed disc, which is a disc with the boundary circle, is an example of a closed set. The whole plane and the empty set are open as well as closed sets. Finite union or intersection of open (respectively, closed) sets are also open (respectively, closed).[4] A connected open set is called a domain.

Here, we summarize the basic notions about differential calculus.

(i) $f(z)$ is called differentiable in a set S if it is differentiable at every point in S.
(ii) $f(z)$ is called holomorphic at a point z if it is differentiable in some neighborhood of z.
(iii) $f(z)$ is called holomorphic in a set S if it is holomorphic at every point in S.
(iv) $u(x, y)$ is called totally differentiable in a set S if it is totally differentiable at every point in S.
(v) $u(x, y)$ is called of class C^1 in a domain D, if u is continuous in D, partially differentiable in D, and u_x and u_y are continuous in D. Similarly, u is called of class C^r in D, if u is continuous in D, r times partially differentiable in D, and all partial derivatives of order at most r are continuous in D. u is called of class C^∞ in D, if it is of class C^r in D for every natural number r.

According to (i) – (iii), the proposition that $f(z)$ is differentiable in a open set or domain D is equivalent to the proposition that $f(z)$ is holomorphic in D. For a general set S, $f(z)$ is holomorphic in S, if it is differentiable in some open set containing S.

By Theorem 2, we have

Theorem 3. *The necessary and sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be holomorphic in a domain D is that u and v are totally differentiable in D and satisfy the Cauchy–Riemann equations in D.*

Here, as we shall study later, if $f(z)$ is holomorphic in a domain D, then $f'(z)$ is also holomorphic in D. Hence $f''(z)$ is again holomorphic in D, and therefore $f(z)$ is differentiable any times in D and its derivatives of any order are continuous in D. Thus by (12)), u and v are partially differentiable any times in D and their partial derivatives are continuous in D. Consequently, u and v are of class C^r in D for any r, and so of class C^∞ in D.

By this argument, we can replace the word "totally differentiable" in Theorem 3 with "of class C^1". This version is convenient because it is easy to determine whether a function is of class C^1 or not.

[4]The union of infinitely many open sets are also open, whereas the intersection of infinitely many open sets are not always open. The intersection of infinitely many closed sets are also closed, whereas the union of infinitely many closed sets are not always closed.

Theorem 3'. *The necessary and sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be holomorphic in a domain D is that u and v are of class C^1 in D and satisfy the Cauchy–Riemann equations in D.*

2.7. **Harmonic functions.** If a function $u(x, y)$ is of class C^2 and u satisfies Laplace's equation:

$$u_{xx} + u_{yy} = 0, \tag{26}$$

then u is called a harmonic function. If two harmonic functions u and v satisfy the Cauchy–Riemann equations, then v is called conjugate to u. The condition for $f(z)$ to be holomorphic is described in terms of harmonic functions.

Theorem 4. *The necessary and sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be holomorphic in a domain D is that u and v are harmonic in D and v is conjugate to u in D.*

Proof. Let $f(z)$ be holomorphic in D. By the description in Section2.6, u and v are of class C^2. Also, as they satisfy the Cauchy–Riemann equations: $u_x = v_y$, $u_y = -v_x$ in D, we have

$$\begin{aligned} u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0 \\ v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0. \end{aligned} \tag{27}$$

Hence u and v are harmonic in D. It is clear that v is conjugate to u in D.

Conversely, if u and v are harmonic and v is conjugate in D, then $f(z)$ is obviously holomorphic in D. □

(exercise03) Show the function $u = x^3 + 3x^2y - 3xy^2 - y^3$ is harmonic, and determine the holomorphic function $f(z) = u + iv$.
(answer)

$$\begin{aligned} u_{xx} + u_{yy} \quad &= (3x^2 + 6xy - 3y^2)_x + (3x^2 - 6xy - 3y^2)_y \\ &= 6x + 6y - 6x - 6y = 0. \end{aligned} \tag{28}$$

Hence u is harmonic. Next as $f(z)$ is holomorphic, by the Cauchy–Riemann equations,

$$u_x = 3x^2 + 6xy - 3y^2 = v_y, \tag{29}$$

$$u_y = 3x^2 - 6xy - 3y^2 = -v_x. \tag{30}$$

By (29),

$$v = \int (3x^2 + 6xy - 3y^2) dy = 3x^2y + 3xy^2 - y^3 + g(x). \tag{31}$$

Substituting this in (30),

$$\begin{aligned} & 3x^2 - 6xy - 3y^2 = -(6xy + 3y^2 + g'(x)). \\ \therefore \quad & g'(x) = -3x^2. \qquad\qquad \therefore \quad g(x) = -x^3 + c \quad (c \in \mathbb{R}). \\ \therefore \quad & f(z) = x^3 + 3x^2y - 3xy^2 - y^3 + i(-x^3 + 3x^2y + 3xy^2 - y^3 + c) \\ & \quad\;\; = (1 - i)z^3 + ic. \end{aligned} \tag{32}$$

2.8. **Limits of functions.** In this chapter, we have dealt with rather simple limits of functions. For the next chapter and beyond, several formulas concerning limits of functions are listed below. For complex functions $f(z)$, $g(z)$, suppose $\lim_{z\to c} f(z)$ and $\lim_{z\to c} g(z)$ exist, then the following holds.

$$\begin{aligned} &\lim_{z\to c}(f(z)\pm g(z)) = \lim_{z\to c} f(z) \pm \lim_{z\to c} g(z) \qquad && \lim_{z\to c} kf(z) = k\lim_{z\to c} f(z) \\ &\lim_{z\to c} f(z)g(z) = \lim_{z\to c} f(z)\cdot \lim_{z\to c} g(z) && \lim_{z\to c}\frac{f(z)}{g(z)} = \frac{\lim_{z\to c} f(z)}{\lim_{z\to c} g(z)} \\ &\lim_{z\to c}\tilde{g}(f(z)) = \tilde{g}\left(\lim_{z\to c} f(z)\right). \end{aligned} \tag{33}$$

Formulas for the sum or product of more than two functions are very similar. For the fourth formula, suppose $\lim_{z\to c} g(z) \neq 0$, and the last formula, $\tilde{g}(w)$ is continuous at $w = \lim_{z\to c} f(z)$.

For a complex function $f(z)$, $\lim_{z\to\infty} f(z)$ means the limit of $f(z)$ as z comes closer to a point of infinity ∞ on the Riemann sphere. In other words, letting $z = 1/t$,

$$\lim_{z\to\infty} f(z) = \lim_{t\to 0} f(1/t). \tag{34}$$

Similarly, $\lim_{z\to c} f(z) = \infty$ means that $f(z)$ comes closer to a point of infinity ∞ as $z\to c$, and so it is defined by

$$\lim_{z\to c} f(z) = \infty \iff \lim_{z\to c}\frac{1}{f(z)} = 0. \tag{35}$$

CHAPTER 3
HOLOMORPHIC FUNCTIONS
⋆ 12 ⋆

KEYWORDS: DIFFERENTIATION FORMULAS, ENTIRE FUNCTIONS, POLYNOMIALS, RATIONAL FUNCTIONS, EXPONENTIAL FUNCTIONS, TRIGONOMETRIC FUNCTIONS, PERIODIC FUNCTIONS, INVERSE FUNCTIONS, LOGARITHMIC FUNCTIONS, ROOT FUNCTIONS, GENERAL POWERS

3.1. **Holomorphy and continuity.** Hereafter, we write "complex functions" simply as "functions" unless confusion occurs. If a function $f(z)$ is differentiable at a point z, then $f(z)$ is continuous at z, because letting $\Delta z \to 0$ in the equality

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = f'(z)+\epsilon, \tag{1}$$

we have $\epsilon \to 0$, and therefore $f(z+\Delta z) = (f'(z)+\epsilon)\Delta z + f(z) \to f(z)$. Hence, a holomorphic function in D is continuous in D. (Of course, the converse does not hold.)

3.2. **Differentiation formulas.** Complex functions satisfy several differentiation formulas similar to ones for real functions. That is, if f and g are holomorphic in a domain D, then $f+g$, cf, fg are also holomorphic in D, and $\frac{f}{g}$ is also holomorphic in D except the points such that $g=0$ and,

$$(f+g)' = f'+g' \qquad (cf)' = cf' \qquad (fg)' = f'g+fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2} \tag{2}$$

Let us confirm, for example the multiplicative case:

$$\begin{array}{rl} & \frac{1}{\Delta z}[f(z+\Delta z)g(z+\Delta z)-f(z)g(z)] \\ = & \frac{1}{\Delta z}[f(z+\Delta z)g(z+\Delta z)-f(z)g(z+\Delta z)+f(z)g(z+\Delta z)-f(z)g(z)] \\ = & \frac{1}{\Delta z}[f(z+\Delta z)-f(z)]\cdot g(z+\Delta z)+f(z)\cdot\frac{1}{\Delta z}[g(z+\Delta z)-g(z)] \\ \longrightarrow & f'(z)g(z)+f(z)g'(z). \qquad (\Delta z\to 0) \qquad \text{(By the continuity of } g) \end{array} \tag{3}$$

(exercise01) (1) Show the rest formulas. (2) Show that $(f_1+f_2+\cdots+f_n)' = f_1'+f_2'+\cdots+f_n'$.

3.3. **Composition and differentiation.** Composite functions satisfy the usual differentiation formulas. Let f be holomorphic in a domain D and g be holomorphic in $f(D)$. Then $g \circ f$ is holomorphic in D, and satisfies the chain rule:

$$(g \circ f)' = (g' \circ f)f', \qquad \text{i.e.} \quad [g(f(z))]' = g'(f(z))f'(z). \tag{4}$$

For, letting $w = f(z)$, $\Delta w = f(z + \Delta z) - f(z)$, as f is continuous in D, $\Delta z \to 0$ implies $\Delta w \to 0$, and

$$\begin{aligned} & \tfrac{1}{\Delta z}[g(f(z+\Delta z)) - g(f(z))] \\ = \quad & \tfrac{1}{\Delta w}[g(w+\Delta w) - g(w)] \cdot \tfrac{1}{\Delta z}[f(z+\Delta z) - f(z)] \\ \longrightarrow \quad & g'(w)f'(z) = g'(f(z))f'(z). \qquad (\Delta z \to 0) \end{aligned} \tag{5}$$

3.4. **Holomorphic functions.** The identity function $f(z) = z$ or a constant function $f(z) = c$ is clearly holomorphic in the whole complex plane. Starting with those functions, iterating addition and multiplication finite times, we have a polynomial, which is holomorphic in the whole complex plane because of the contents of 3.2. Furthermore, division of polynomials makes a rational function, which is holomorphic in the whole plane except the points where the denominator vanishes. A holomorphic function in the whole complex plane is called an entire function. A point where a function is not holomorphic is called a singularity or a singular point. A singularity is usually marked by $\times$.
(exercise02) Show $(z^n)' = nz^{n-1}$ by induction.

3.5. **The exponential function.** Here, we introduce the exponential function e^z as an important example of a non-polynomial entire function. We define e^z by

$$e^z = e^{x+yi} := e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y. \tag{6}$$

Letting $e^z = u(x,y) + iv(x,y)$, we see that u and v are of class C^1 and satisfy Cauchy–Riemann equations in the whole complex plane. Hence e^z is an entire function. Obviously, e^z coincides with the usual exponential function when z is a real number. If $f(z)$ is a long expression, we sometimes write $e^{f(z)} = \exp(f(z))$.

The exponential function satisfies the following formulas.

$$\boxed{\begin{array}{c} (e^z)' = e^z \\ e^{z+w} = e^z e^w \text{ (an exponential law)} \qquad e^{z+2\pi i} = e^z \\ |e^z| = e^{\operatorname{Re} z} \qquad \arg e^z = \operatorname{Im} z + 2n\pi \ (n \in \mathbb{Z}) \\ \overline{e^z} = e^{\overline{z}} \end{array}} \tag{7}$$

(exercise03) (1) Confirm that e^z is an entire function. (2) Show (7). (3) Confirm that $e^0 = 1$, $e^{-z} = \frac{1}{e^z}$, $e^{z-w} = \frac{e^z}{e^w}$.
(exercise04) Show that for any $z \in \mathbb{C}$, $e^z \neq 0$.

3.6. **The trigonometric functions.** The trigonometric functions are defined using the exponential function as follows.

$$\begin{array}{lll} \cos z = \frac{e^{iz}+e^{-iz}}{2} & \sin z = \frac{e^{iz}-e^{-iz}}{2i} & \tan z = \frac{\sin z}{\cos z} \\ \sec z = \frac{1}{\cos z} & \csc z = \frac{1}{\sin z} & \cot z = \frac{\cos z}{\sin z} \end{array} \tag{8}$$

Beside these functions, the hyperbolic functions below are used.

$$\begin{array}{lll} \cosh z = \frac{e^{z}+e^{-z}}{2} & \sinh z = \frac{e^{z}-e^{-z}}{2} & \tanh z = \frac{\sinh z}{\cosh z} \\ \operatorname{sech} z = \frac{1}{\cosh z} & \operatorname{csch} z = \frac{1}{\sinh z} & \coth z = \frac{\cosh z}{\sinh z} \end{array} \tag{9}$$

By the argument in 3.2 – 3.5, every function expressed by the exponential function and addition and multiplication is an entire function. Hence, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions. All those functions we defined in this section coincide with the usual ones when z is a real number. The trigonometric functions satisfy the formulas below.

$$\boxed{\begin{array}{c} (\cos z)' = -\sin z \qquad (\sin z)' = \cos z \\ \cos(z+w) = \cos z \cos w - \sin z \sin w \\ \sin(z+w) = \sin z \cos w + \cos z \sin w \\ \cos(z+2\pi) = \cos z \quad \sin(z+2\pi) = \sin z \quad \tan(z+\pi) = \tan z \\ \cos^2 z + \sin^2 z = 1 \\ e^{iz} = \cos z + i \sin z \end{array}} \tag{10}$$

The hyperbolic functions satisfy the formulas below.

$$\boxed{\begin{array}{c} (\cosh z)' = \sinh z \qquad (\sinh z)' = \cosh z \\ \cosh iz = \cos z \qquad \sinh iz = i \sin z \\ \cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w \\ \sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w \\ \cosh(z+2\pi i) = \cosh z \qquad \sinh(z+2\pi i) = \sinh z \\ \tanh(z+\pi i) = \tanh z \\ \cosh^2 z - \sinh^2 z = 1 \end{array}} \tag{11}$$

(exercise05) Show that $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$.
(exercise06) Show that $\cos(z+w) = \cos z \cos w - \sin z \sin w$, $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
(exercise07) (1) Show that $\sin^2 z + \cos^2 z = 1$. (2) Show that $e^{iz} = \cos z + i \sin z$.

3.7. **Periodic functions.** For a function $f(z)$, if there exists a nonzero constant ω, such that, for every point z in the domain of $f(z)$, $z+\omega$ also belongs to the domain, and it holds that

$$f(z+\omega) = f(z), \tag{12}$$

then $f(z)$ is called a periodic function, and ω is called the period of $f(z)$. As we have seen in 3.5, 3.6, the exponential function, the trigonometric functions and the hyperbolic functions are periodic functions. For the exponential function and the trigonometric functions, we have

$$\begin{array}{rcll} e^{z+\omega} = e^z & \Longleftrightarrow & \omega = 2n\pi i & (n \in \mathbb{Z}) \\ \cos(z+\omega) = \cos z & \Longleftrightarrow & \omega = 2n\pi & (n \in \mathbb{Z}) \\ \sin(z+\omega) = \sin z & \Longleftrightarrow & \omega = 2n\pi & (n \in \mathbb{Z}). \end{array} \tag{13}$$

Hence the periods of e^z are $2n\pi$ only, and the periods of $\cos z$ or $\sin z$ are $2n\pi$ only. A table of the periods of important periodic functions follows.

e^z	$2n\pi i$
$\cos z$, $\sin z$	$2n\pi$
$\tan z$	$n\pi$
$\cosh z, \sinh z$	$2n\pi i$
$\tanh z$	$n\pi i$

3.8. **Inverse functions.** For a function $f(z)$, a function $w = g(z)$ given by solving an equation $f(w) = z$ with respect to w is called the inverse function of $f(z)$. This definition of the inverse function $g(z)$ is equivalent to

$$f(w) = z \iff w = g(z). \tag{14}$$

The inverse function of a complex function is sometimes multivalued[1], but by definition, it always holds that $f(g(z)) = z$. The inverse function of f is denoted also by f^{-1}.

Theorem 1. *Let D be a domain, let $f : D \longrightarrow E$ be a holomorphic surjection*[2]*, and let the inverse (if multivalued, take a single-valued branch of them* [3]*) $g : E \longrightarrow D$ of f be continuous. Suppose $f'(w) \neq 0$ in D. Then g is holomorphic in the interior of E and $g'(z) = \frac{1}{f'(g(z))}$.*

Proof. Take an arbitrary point z in the interior of E and fix it. Letting $w = g(z)$, we have $w \in D$ and $f(w) = z$. When z moves by Δz, suppose w moves by Δw. Since g is continuous, $\Delta z \to 0$ implies $\Delta w \to 0$. Therefore

$$\lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta w \to 0} \frac{1}{\frac{\Delta z}{\Delta w}} = \frac{1}{f'(w)} = \frac{1}{f'(g(z))} = g'(z), \tag{15}$$

and so g is differentiable at z. Hence g is holomorphic in the interior of E. □

[1]If a function has finitely or infinitely many values at a point, then it is called a multivalued function. Regarding each distinct value as a distinct function, a multivalued function is considered also as a set of functions. An ordinary function with one value at every point is called a single-valued function. If a function has up to n values in its domain, then it is called an n-valued function. A function not n-valued for finite n is called an infinitely many-valued function.

[2]f is holomorphic in D and satisfies $f(D) = E$.

[3]A single-valued function made by taking a value of multivalued function is called a single-valued branch of it.

3.9. **The logarithmic function.** The inverse function of e^z is called the logarithmic function denoted by $\log z$. That is, if we solve $e^w = z$ with respect to w, then we have $w = \log z$. Letting $w = u + iv$,

$$e^w = e^{u+iv} = e^u e^{iv} = z. \tag{16}$$

Here taking the absolute values of both sides, we have $e^u = |z| \iff u = \mathrm{Log}\,|z|$, where $\mathrm{Log}\,|z|$ denotes the natural logarithm of the positive numbers. Next comparing the arguments of both sides, we have $v = \arg z$. Consequently,

$$w = \log z = \mathrm{Log}\,|z| + i \arg z. \tag{17}$$

Since arg is multivalued, log is also multivalued. The argument of z satisfying $-\pi < \theta \le \pi$ is called the principal value of the argument, denoted by $\mathrm{Arg}\, z$. This is uniquely determined if $z \neq 0$. Using this value, we have

$$\log z = \mathrm{Log}\,|z| + i\mathrm{Arg}\, z + 2n\pi i \qquad (n \in \mathbb{Z}). \tag{18}$$

In addition, the function

$$\mathrm{Log}\, z = \mathrm{Log}\,|z| + i\mathrm{Arg}\, z \tag{19}$$

is called the principal value of the logarithm. From this, it follows that

$$\log z = \mathrm{Log}\, z + 2n\pi i. \tag{20}$$

Though the logarithmic function is multivalued (infinitely many-valued), if we note its one value (single-value branch), by (17), it is continuous in a simply-connected[4] domain. Also, it always holds that $(e^w)' = e^w \neq 0$. Hence by Theorem 1, (a single-value branch of) the logarithmic function is holomorphic in such a domain, and its derivative is given by

$$(\log z)' = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}. \tag{21}$$

As for the (complex) logarithm, ordinary formulas for the real logarithm sometimes fail. It is valid that

$$\begin{array}{l} \log(z_1 z_2) = \log z_1 + \log z_2, \quad \log \frac{z_1}{z_2} = \log z_1 - \log z_2, \quad \log \frac{1}{z} = -\log z, \\ \log z^c = c \log z + 2n\pi i \quad (c \in \mathbb{C}) \qquad (\mathit{See}\ 3.12), \\ \log z^{1/m} = \frac{1}{m} \log z \quad (m \in \mathbb{Z},\, m \neq 0) \qquad (\mathit{See}\ 3.12), \end{array} \tag{22}$$

but some formulas like $\log z^2 = 2 \log z$ are not valid. The meaning of (22) is that both sides are equal as multivalued functions, that is, the set of values corresponding to every z are equal. For $\log z^2 = 2 \log z$, there are common values in both sides, but the sets of values in both sides are not identical, and the formula fails.

Proof of (22) *(the first identity).* Letting $w_1 = \mathrm{Log}\, z_1$, $w_2 = \mathrm{Log}\, z_2$, we have $e^{w_1} = z_1$, $e^{w_2} = z_2$. $\therefore$ $e^{w_1 + w_2} = z_1 z_2$. $\therefore$ $\log(z_1 z_2) = w_1 + w_2 + 2n\pi i$ (#1). On the other hand, $\log z_1 + \log z_2 = w_1 + 2n\pi i + w_2 + 2m\pi i = w_1 + w_2 + 2(m+n)\pi i$ (#2). Since (#1) and (#2) coincide with each other as the sets of values, we have $\log(z_1 z_2) = \log z_1 + \log z_2$. $\square$

[4] A domain without hole is called simply-connected domain. If a point is excluded from a domain, it is not simply-connected.

3.10. **Radical functions.** The inverse function of z^n is called a radical function, debited by $\sqrt[n]{z}$. It is obtained by solving $w^n = z$. Letting $w = se^{i\varphi}$,

$$s^n e^{in\varphi} = z. \tag{23}$$

Taking the absolute values, $s^n = |z|$, $s = \sqrt[n]{|z|}$. Comparing the arguments, $n\varphi = \arg z$. $\varphi = \frac{\arg z}{n} = \frac{\operatorname{Arg} z + 2m\pi}{n}$ $(m = 0, \ldots, n-1)$. Therefore

$$w = \sqrt[n]{z} = \sqrt[n]{|z|} e^{i\frac{\arg z}{n}} = \sqrt[n]{|z|} e^{i\frac{\operatorname{Arg} z + 2m\pi}{n}} \quad (m = 0, \ldots, n-1). \tag{24}$$

Hence $\sqrt[n]{z}$ is multivalued (n-valued). Here, $\sqrt[n]{|z|} e^{i\frac{\operatorname{Arg} z}{n}}$ is called the principal value of $\sqrt[n]{z}$.

A (single-valued branch of a) radical function is also holomorphic in a simply-connected domain not containing $z = 0$, its derivative is given by

$$\left(\sqrt[n]{z}\right)' = \frac{1}{(w^n)'} = \frac{1}{nw^{n-1}} = \frac{1}{n\left(\sqrt[n]{z}\right)^{n-1}}. \tag{25}$$

3.11. **Other inverse functions.** The inverse functions of trigonometric functions and hyperbolic functions are denoted by attaching "arc" at the beginning of the function names. For example, the inverse of $\sin z$ is denoted by $\arcsin z$, etc. In general, they are represented by using logarithms, and thus multivalued. We have

$$\begin{aligned} &1: \ \arccos z = i \log\left(z + \sqrt{z^2 - 1}\right) \quad 2: \ \arcsin z = i \log\left(-iz + \sqrt{1 - z^2}\right) \\ &3: \ \arctan z = \tfrac{i}{2} \log \tfrac{i+z}{i-z} \end{aligned} \tag{26}$$

(exercise08) Show those formulas.
(answer) We show 1:. Solve $\cos w = z$ with respect to w.

$$\begin{aligned} &\cos w = \tfrac{e^{iw} + e^{-iw}}{2} = z. \qquad \text{Here letting} e^{-iw} = s, \\ &s^{-1} + s = 2z. \qquad \therefore \ s^2 - 2zs + 1 = 0. \qquad \therefore \ s = z + \sqrt{z^2 - 1}. \\ &\therefore \ \ w = \arccos z = i \log s = i \log\left(z + \sqrt{z^2 - 1}\right). \end{aligned} \tag{27}$$

3.12. **General powers.** For complex numbers z, c $(z \neq 0)$, define z^c by

$$z^c = e^{c \log z}. \tag{28}$$

If c is an integer, then z^c is single-valued, which coincides with the ordinary c-th power of z, whereas if c is not an integer, then z^c is multivalued. We call $e^{c \operatorname{Log} z}$ the principal value of z^c. In this text, denote the principal value of z^c by $(z^c)_{\mathrm{pv}}$. We have the following.

$$\begin{aligned} &z^c = (z^c)_{\mathrm{pv}}\, e^{2cn\pi i}, \quad (z_1 z_2)^c = z_1^c z_2^c, \quad \left(\tfrac{z_1}{z_2}\right)^c = \tfrac{z_1^c}{z_2^c}, \quad z^{-c} = \tfrac{1}{z^c} = \left(\tfrac{1}{z}\right)^c, \\ &\left(z^{c_1 + c_2}\right)_{\mathrm{pv}} = \left(z^{c_1}\right)_{\mathrm{pv}} \left(z^{c_2}\right)_{\mathrm{pv}} \end{aligned} \tag{29}$$

Proof of the first and the second formula.

$$\begin{aligned} z^c &= e^{c \log z} = e^{c(\operatorname{Log} z + 2n\pi i)} = e^{c \operatorname{Log} z} e^{2cn\pi i} = (z^c)_{\mathrm{pv}}\, e^{2cn\pi i}. \\ (z_1 z_2)^c &= e^{c \log(z_1 z_2)} = e^{c(\log z_1 + \log z_2)} = e^{c \log z_1 + c \log z_2} \\ &= e^{c \log z_1} e^{c \log z_2} = z_1^c z_2^c. \quad \square \end{aligned} \tag{30}$$

(note) Some formulas such as (i) $z^{c_1+c_2} = z^{c_1}z^{c_2}$, (ii) $z^{c_1c_2} = (z^{c_1})^{c_2}$ are not valid in general. They have common values on both sides, but not identical as the sets of values. (i) is calculated as

$$\begin{aligned} z^{c_1+c_2} &= \left(z^{c_1+c_2}\right)_{\mathrm{pv}} e^{2(c_1+c_2)n\pi i}, \\ z^{c_1}z^{c_2} &= \left(z^{c_1}\right)_{\mathrm{pv}} e^{2c_1n\pi i}\left(z^{c_2}\right)_{\mathrm{pv}} e^{2c_2n'\pi i} = \left(z^{c_1+c_2}\right)_{\mathrm{pv}} e^{2c_1n\pi i+2c_2n'\pi i}. \qquad (n, n' \in \mathbb{Z}) \end{aligned} \tag{31}$$

Regarding those as the set of values, we see that the left-hand side is included in the right-hand side. Now for example, letting $c_1 = m$ be an integer, $c_2 = c$ be a complex number, we have

$$z^{m+c} = z^m z^c. \tag{32}$$

For (ii), we have

$$z^{c_1c_2} = \left(z^{c_1c_2}\right)_{\mathrm{pv}} e^{2c_1c_2n\pi i}, \quad (z^{c_1})^{c_2} = \left(z^{c_1c_2}\right)_{\mathrm{pv}} e^{2c_1c_2n\pi i}e^{2c_2n'\pi i}, \tag{33}$$

and again the left-hand side is included in the right-hand side. In this case, letting $c_1 = c$ be a complex number and $c_2 = m$ be an integer, we have

$$z^{cm} = (z^c)^m \tag{34}$$

(exercise09) Calculate the following. (1) $\mathrm{Log}\,(-ei)$. $\left(1 - \frac{\pi}{2}i\right)$ (2) $\left(1 + \sqrt{3}i\right)^i$. $\left(e^{-\frac{\pi}{3}-2n\pi+i\mathrm{Log}\,2}\right)$

(exercise10) Show that $z^{\frac{1}{n}} = \sqrt[n]{z}$.

(answer)

$$\begin{aligned} z^{\frac{1}{n}} &= e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\mathrm{Log}\,|z|+i\arg z)} = e^{\frac{1}{n}\mathrm{Log}\,|z|}e^{i\frac{\arg z}{n}} \\ &= e^{\mathrm{Log}\,|z|^{\frac{1}{n}}}e^{i\frac{\arg z}{n}} = |z|^{\frac{1}{n}}e^{i\frac{\arg z}{n}} = \sqrt[n]{|z|}e^{i\frac{\arg z}{n}} = \sqrt[n]{z}. \end{aligned} \tag{35}$$

(exercise11) Let $\frac{q}{p}$ be an irreducible fraction and $p \geq 1$. Show that $z^{\frac{q}{p}}$ has exactly p distinct values.

(answer) Letting $\mathrm{Arg}\,z = \theta$,

$$\begin{aligned} z^{\frac{q}{p}} &= e^{\frac{q}{p}\log z} = e^{\frac{q}{p}(\mathrm{Log}\,|z|+i\arg z)} = e^{\frac{q}{p}\mathrm{Log}\,|z|}e^{i\frac{q}{p}\arg z} \\ &= e^{\mathrm{Log}\,|z|^{\frac{q}{p}}}e^{i\frac{q}{p}(\theta+2m\pi)} = |z|^{\frac{q}{p}}e^{i\frac{q}{p}\theta}e^{i\frac{2qm}{p}\pi} \end{aligned} \tag{36}$$

Obviously, (36) takes the same value whenever m increases by p. Hence it suffices to show that (36) has distinct values for $m = 0, 1, 2, \ldots, p-1$. Suppose, for $0 \leq m < m' \leq p-1$, that

$$e^{i\frac{2qm}{p}\pi} = e^{i\frac{2qm'}{p}\pi}, \tag{37}$$

then for some integer k,

$$\frac{2qm'}{p} - \frac{2qm}{p} = \frac{2q(m'-m)}{p} = 2k. \tag{38}$$

This is contradiction, and the proposition is proved. □

(note) Hereafter, the term function or the symbol such as $f(z)$ represents a single-valued function (or a function made by choosing one value from a multivalued function), unless specifically stated otherwise.

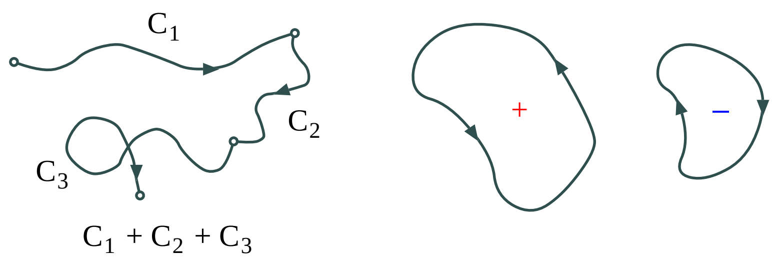
CHAPTER 4

COMPLEX INTEGRALS AND LINE INTEGRALS

⋆ 4 ⋆

KEYWORDS: CURVES, COMPLEX INTEGRALS, LINE INTEGRALS, ESTIMATION LEMMA, ML INEQUALITY

4.1. **Curves.** While a curve C on the complex plane is expressed by some equation of z, it is also expressed by a complex-valued function of a real variable t, that is, $z = z(t) = x(t) + iy(t)$ $(a \leq t \leq b)$. This expression is called a parametric expression of a curve C. Here, $x(t)$ and $y(t)$ are continuous real functions. For the curve C, $\alpha = z(a)$ is called the initial point of C, and $\beta = z(b)$ is called the terminal point of C, and is considered to be oriented from the initial point to the terminal point, called a curve from α to β. A curve with distinct initial and terminal points is called open, while one with identical initial and terminal points is called closed. For any distinct t_1 and t_2 ($\{t_1, t_2\} \neq \{a, b\}$), if C satisfies $z(t_1) \neq z(t_2)$, then C is called a simple curve or a Jordan curve. Intuitively speaking, a simple curve is a curve which does not cross with itself. A simple closed curve C divides the complex plane into the interior and the exterior of C. In addition, we can determine whether a simple closed curve is oriented clockwise or anti-clockwise. Anti-clockwise orientation is defined to be positive, and clockwise orientation is defined to be negative. For a curve C, the curve made from C by changing the orientation inversely is denoted by $-C$.

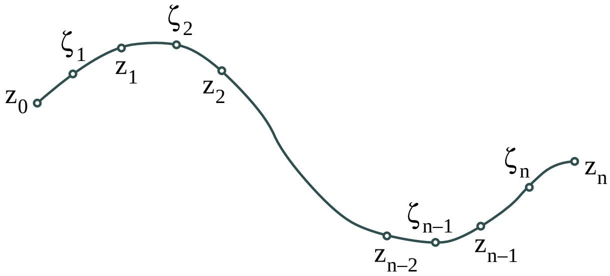


If there exist two curves C_1 and C_2, and the terminal point of C_1 coincides the initial point of C_2, then we can join C_1 and C_2 naturally, and the resulting curve is denoted by $C_1 + C_2$. Iterating this operation, we can join more than two curves under the above-mentioned condition. For a curve $C : z = z(t) = x(t) + y(t)$, if both of $x(t)$ and $y(t)$ are of class C^1, then C is called a smooth curve. A curve formed by joining finitely many smooth curves is called a piecewise smooth curve.

(exercise01) Express a positively oriented circle $|z-c|=r$ in the form $z=z(t)$. $[z=c+re^{it}\ (0\le t\le 2\pi)]$

4.2. **Complex integrals.** Let $f(z)$ be a continuous complex function defined in a domain D. Let $C: z=z(t)=x(t)+iy(t)$ $(a\le t\le b)$ be a smooth curve from α to β in D. Now we introduce the complex integral $\int_C f(z)dz$ of $f(z)$ along a curve C.

Take several points $\alpha=z_0,z_1,\ldots,z_n=\beta$ on C in this order. Further, take a point ζ_k on every interval on C from z_{k-1} to z_k. Then the sequence of points $z_0,\zeta_1,z_1,\zeta_2,z_2,\ldots,\zeta_n,z_n$ is called a partition of C, denoted simply by Δ. The size $|\Delta|$ of the partition Δ is defined to be $|\Delta|=\max_{1\le k\le n}|z_k-z_{k-1}|$. Denote $z_k-z_{k-1}=\Delta z_k$, and let $z_k=x_k+iy_k$, $\Delta z_k=\Delta x_k+i\Delta y_k$.



For a partition Δ of C, consider the following quantity:

$$S_\Delta=S_\Delta(f)=\sum_{k=1}^{n}f(\zeta_k)\Delta z_k. \tag{1}$$

This takes various values depending on a partition, however, we show that it converges to some limit as $|\Delta|\to 0$. First of all, letting $z(t_k)=z_k$ and $z(\tau_k)=\zeta_k$ on $C: z=z(t)$, we have a partition $\tilde{\Delta}$: $a=t_0,\tau_1,t_1,\ldots,\tau_n,t_n=b$ the interval $[a,b]$. Here, we assume that $|\Delta|\to 0$ implies $|\tilde{\Delta}|\to 0$.

$$\begin{aligned} S_\Delta &= \sum_{k=1}^{n}f(\zeta_k)\Delta z_k=\sum_{k=1}^{n}[u(\zeta_k)+iv(\zeta_k)][\Delta x_k+i\Delta y_k] \\ &= \sum_{k=1}^{n}[u(\zeta_k)\Delta x_k-v(\zeta_k)\Delta y_k+iu(\zeta_k)\Delta y_k+iv(\zeta_k)\Delta x_k] \\ &= \sum_{k=1}^{n}u(\zeta_k)\Delta x_k-\sum_{k=1}^{n}v(\zeta_k)\Delta y_k+i\sum_{k=1}^{n}u(\zeta_k)\Delta y_k+i\sum_{k=1}^{n}v(\zeta_k)\Delta x_k \end{aligned} \tag{2}$$

Here, the last expression is the sum of four terms, and for convenience, we calculate the first one,

$$\begin{aligned}
&\sum_{k=1}^{n} u(\zeta_k)\Delta x_k = \sum_{k=1}^{n} u(z(\tau_k))[x(t_k) - x(t_{k-1})] \\
&= \sum_{k=1}^{n} u(z(\tau_k))\frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} u(z(\tau_k))x'(\sigma_k)\Delta t_k \qquad\qquad \text{(By the mean value theorem)} \\
&= \sum_{k=1}^{n} u(z(\tau_k))x'(\tau_k)\Delta t_k + \sum_{k=1}^{n} u(z(\tau_k))[x'(\sigma_k) - x'(\tau_k)]\Delta t_k,
\end{aligned} \tag{3}$$

where $t_k - t_{k-1} = \Delta t_k$. Since $|\Delta| \to 0$ implies $|\tilde{\Delta}| \to 0$, by the definition of integrals of real functions, we have

$$\sum_{k=1}^{n} u(z(\tau_k))x'(\tau_k)\Delta t_k \longrightarrow \int_a^b u(z(t))x'(t)dt \quad (|\Delta| \to 0) \tag{4}$$

Next consider the second term. Since $x'(t)$ is continuous on the closed interval $[a, b]$, it is uniformly continuous[1] on $[a, b]$. Hence for any $\epsilon > 0$, taking sufficiently small $\delta > 0$, it holds that $|\Delta| < \delta$ implies $|x'(\sigma_k) - x'(\tau_k)| < \epsilon$. Then letting the maximum value of $|u(z)|$ on C is M, we have

$$\left|\sum_{k=1}^{n} u(z(\tau_k))[x'(\sigma_k) - x'(\tau_k)]\Delta t_k\right| < M\epsilon(b-a). \tag{5}$$

Therefore as $|\Delta| \to 0$,

$$\sum_{k=1}^{n} u(\zeta_k)\Delta x_k \longrightarrow \int_a^b u(z(t))x'(t)dt. \tag{6}$$

Repeating the above argument,

$$\begin{aligned}
S_\Delta \longrightarrow \quad & \int_a^b u(z(t))x'(t)dt - \int_a^b v(z(t))y'(t)dt \\
& +i\int_a^b u(z(t))y'(t)dt + i\int_a^b v(z(t))x'(t)dt \\
= \quad & \int_a^b [u(z(t)) + iv(z(t))][x'(t) + iy'(t)]dt \\
= \quad & \int_a^b f(z(t))z'(t)dt.
\end{aligned} \tag{7}$$

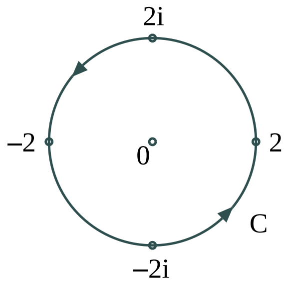
[1]For any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon$, then $g(x)$ is called uniformly continuous. If a function is continuous on the (finite) closed interval, then it is uniformly continuous there.

This limit is denoted by $\int_C f(z)dz$, called the complex integral of $f(z)$ along C. By the definition of S_Δ, $\int_C f(z)dz$ is determined by the function $f(z)$ and the curve C itself, independent of the parametric expression of C. The curve C is called the contour of this complex integral.

The integral (6) is written as $\int_C u(x,y)dx$. Similarly, the limit of the second term of the last expression in (2) is written as $-\int_C v(x,y)dy$. By using this notation, we can transform a complex integral formally as:

$$\begin{aligned}\int_C f(z)dz &= \int_C (u(x,y)+iv(x,y))(dx+idy)\\ &= \int_C u(x,y)dx - \int_C v(x,y)dy + i\int_C u(x,y)dy + i\int_C v(x,y)dx\end{aligned} \tag{8}$$

(exercise02) Let C be $|z|=2$ (positive orientation), then calculate $\int_C \frac{dz}{z}$.



(answer) Since $C : z = 2e^{it}$ $(0 \le t \le 2\pi)$, we have

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{z'(t)}{2e^{it}}dt = \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}}dt = \int_0^{2\pi} idt = [it]_0^{2\pi} = \boxed{2\pi i}. \tag{9}$$

4.3. **Complex integrals along piecewise smooth curves.** We have studied complex integrals along smooth curve C. More generally, if $C = C_1 + \cdots + C_s$ is piecewise smooth and C_k is smooth for $k = 1, \ldots, s$, define

$$\int_C f(z)dz = \sum_{k=1}^{s} \int_{C_k} f(z)dz. \tag{10}$$

The right-hand side is considered as the limit of S_Δ for a partition Δ including all joints of curves. Hence, there is no essential difference between the smooth case and the piecewise smooth case. In addition, no matter how C is decomposed into curves as $C = C_1 + \cdots + C_s$ (allowed division at smooth point), (10) holds. For, consider a partition Δ of C including all joints of curves and all non-smooth points, and assign it to the partitions $\Delta_1, \ldots, \Delta_s$ of the decomposed curves $C_1, \ldots, C_s$, then we have

$$S_\Delta = S_{\Delta_1} + \cdots + S_{\Delta_s}. \tag{11}$$

Letting $|\Delta| \to 0$, $|\Delta_1|, \ldots, |\Delta_s| \to 0$ and we have (10).

It should be noted that, in (1), if the orientation of C is reversed, then the sign of Δz_k clearly changes, and therefore $\displaystyle\int_{-C} f(z)dz = -\int_C f(z)dz$.

For continuous functions $f(z), g(z)$, from (1) it follows that

$$\begin{aligned} S_\Delta(f+g) = & \sum_{k=1}^{n}(f(\zeta_k)+g(\zeta_k))\Delta z_k = \sum_{k=1}^{n} f(\zeta_k)\Delta z_k + \sum_{k=1}^{n} g(\zeta_k)\Delta z_k \\ = & S_\Delta(f)+S_\Delta(g). \end{aligned} \tag{12}$$

Similarly, for a complex constant m, it holds that $S_\Delta(mf) = mS_\Delta(f)$. Letting $|\Delta| \to 0$, we have the following formulas.

$$\begin{aligned} &\int_C (f(z)+g(z))\,dz = \int_C f(z)dz + \int_C g(z)dz \\ &\int_C mf(z)dz = m\int_C f(z)dz \qquad (m \in \mathbb{C}) \end{aligned} \tag{13}$$

(For the first formula, the similar formula holds for more than two summands) This is derived also from (7).

4.4. **Line integral with respect to arc length.** For $f(z)$, C, a partition Δ in 4.2, consider

$$T_\Delta = T_\Delta(f) = \sum_{k=1}^{n} f(\zeta_k)|\Delta z_k|, \tag{14}$$

and the limit of this quantity is called a line integral with respect to arc length, denoted by $\displaystyle\int_C f(z)|dz|$. By definition, several formulas similar to (10),(13) hold for this integral, except for $\displaystyle\int_{-C} f(z)|dz| = \int_C f(z)|dz|$, say, this integral is independent of the orientation of a curve. Also, note that if the length of C be L, then $\displaystyle\int_C |dz| = L$ for $f(z) = 1$. (This is rather the definition of the length of a curve.)

If a curve $C : z = z(t)$ $(a \le t \le b)$ is given, in a similar method as for S_Δ, the limit of T_Δ is calculated (omitted details) as

$$\int_C f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt. \tag{15}$$

This integral is often used for the evaluation of $\displaystyle\left|\int_C f(z)dz\right|$.

Theorem 1. *(The estimation lemma, ML inequality) Let $f(z)$ be continuous function and let C be a curve of length L, and let the maximum value of $|f(z)|$ on C be M, then*

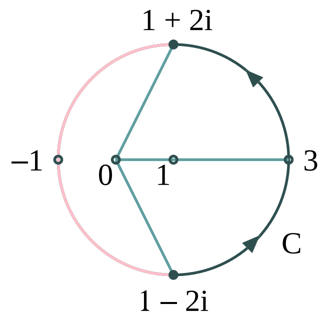
$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le ML. \tag{16}$$

Proof. From $|z+w| \leq |z|+|w|$, it follows that, in general, $|z_1+\cdots+z_n| \leq |z_1|+\cdots+|z_n|$. Hence,

$$\left|\sum_{k=1}^{n} f(\zeta_k)\Delta z_k\right| \leq \sum_{k=1}^{n} |f(\zeta_k)||\Delta z_k| \leq \sum_{k=1}^{n} M|\Delta z_k|. \tag{17}$$

Here, letting $|\Delta| \to 0$, we have (16). □

(exercise03) Let C be a curve along the right half of the circle $|z-1| = 2$, from $1-2i$ to $1+2i$. Show that $\left|\int_C \frac{\overline{z}}{z-1} dz\right| \leq 3\pi$.



(answer) From this figure, we have, on C,

$$\begin{aligned} \left|\frac{\overline{z}}{z-1}\right| &= \frac{|\overline{z}|}{|z-1|} = \frac{|z|}{|z-1|} \leq \frac{3}{2}. \\ \therefore \quad \left|\int_C \frac{\overline{z}}{z-1} dz\right| &\leq \frac{3}{2} \cdot 2\pi = 3\pi. \quad \square \end{aligned} \tag{18}$$

(note) Hereafter, as a contour or the boundary of a figure, we use only piecewise smooth curves unless specifically stated otherwise.

CHAPTER 5

CAUCHY'S INTEGRAL THEOREM AND INTEGRAL FORMULA

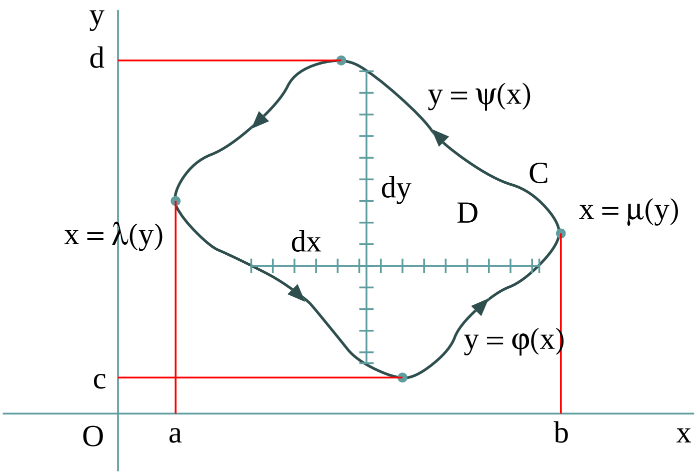
⋆ 8 ⋆

KEYWORDS: GREEN'S THEOREM, CAUCHY'S INTEGRAL THEOREM, PRINCIPLE OF DEFORMATION OF CONTOURS, PRIMITIVE FUNCTIONS, CAUCHY'S INTEGRAL FORMULA

5.1. **Green's theorem.** Let C be a positively oriented simple closed curve,[1] let D be the set consisting of C and the interior of C. Let $P(x, y)$ and $Q(x, y)$ be two-variable real functions of class C^1.

Theorem 1. *(Green's theorem) We have*

$$\int_C (Pdx + Qdy) = \int_D (-P_y + Q_x)dxdy. \tag{1}$$



Proof. For simplicity, C is as depicted below, and the upper half of C is expressed as $y = \psi(x)$ $(a \leq x \leq b)$, and the lower half, $y = \varphi(x)$ $(a \leq x \leq b)$; also, the left half is expressed as $x = \lambda(y)$ $(c \leq y \leq d)$, and the right half, $x = \mu(y)$ $(c \leq y \leq d)$. We have

[1]This theorem is usually formulated on the coordinate plane, but it is still valid on the complex plane. Hereafter, we use curves, domains or figures in the complex plane unless specifically stated otherwise.

$$
\begin{aligned}
\int_D (-P_y + Q_x) dxdy &= \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} - P_y dy \right) dx + \int_c^d \left(\int_{\lambda(y)}^{\mu(y)} Q_x dx \right) dy \quad \text{(Fubini)} \\
&= \int_a^b \left[-P(x,y) \right]_{y=\varphi(x)}^{y=\psi(x)} dx + \int_c^d \left[Q(x,y) \right]_{x=\lambda(y)}^{x=\mu(y)} dy \\
&= \int_a^b \left[-P(x,\psi(x)) + P(x,\varphi(x)) \right] dx \\
&\quad + \int_c^d \left[Q(\mu(y),y) - Q(\lambda(y),y) \right] dy \\
&= \int_C P(x,y)dx + \int_C Q(x,y)dy \quad \text{(By definition)} \\
&= \int_C (P(x,y)dx + Q(x,y)dy).
\end{aligned} \tag{2}
$$

If C is more complicated, then we decompose D by several horizontal or vertical lines, and apply this result to each decomposed part to prove this theorem. □

5.2. **Cauchy's integral theorem I.** Let C be a simple closed curve, $f(z)$ be a complex function holomorphic on and in the interior of C.

Theorem 2. *(Cauchy's integral theorem I) We have*

$$\int_C f(z)dz = 0. \tag{3}$$

Proof. We may suppose the orientation of C is positive. Let D be the set consisting of C and the interior of C.

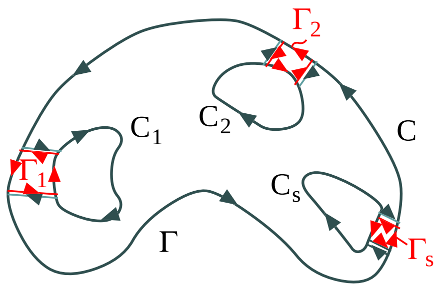
$$
\begin{aligned}
\int_C f(z)dz &= \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i \int_C (vdx + udy) \\
&= \int_D (-u_y - v_x) dxdy + i \int_D (-v_y + u_x) dxdy \quad \text{(Green)} \\
&= \int_D (v_x - v_x) dxdy + i \int_D (-v_y + v_y) dxdy \quad \text{(Cauchy–Riemann)} \\
&= 0 \quad \square
\end{aligned} \tag{4}
$$

This is an important theorem, every forthcoming theorem is derived on the basis of it. Usually it is called "Cauchy's integral theorem", however, to distinguish with the following corollary (Theorem 3), we added the number to the name.

5.3. **Cauchy's integral theorem II.** Let C be a simple closed curve, and $C_1, C_2, \ldots,$ C_s be simple closed curves in C, which are in the exterior of each other. Suppose all closed curves are positively oriented. Suppose $f(z)$ is holomorphic on C and on $C_1, C_2, \ldots, C_r$, and in the domain between these curves.

Theorem 3. *(Cauchy's integral theorem II) We have*

$$\int_C f(z)dz = \sum_{k=1}^{s} \int_{C_k} f(z)dz. \tag{5}$$



Proof. Make two paths between C_k and C for every k as in the above figure. Let Γ be a closed curve obtained by joining C and $C_1, C_2, \ldots, C_s$. Let Γ_k be a small closed curve obtained by joining two paths between C_k and C. Then $f(z)$ is holomorphic on and in the interior of Γ, and also holomorphic on and in the interior of Γ_k. Hence by Cauchy's integral theorem I,

$$\begin{aligned} \int_\Gamma f(z)dz + \sum_{k=1}^{s} \int_{\Gamma_k} f(z)dz &= \int_C f(z)dz - \sum_{k=1}^{s} \int_{C_k} f(z)dz = 0. \\ \therefore \qquad \int_C f(z)dz &= \sum_{k=1}^{s} \int_{C_k} f(z)dz. \quad \square \end{aligned} \tag{6}$$

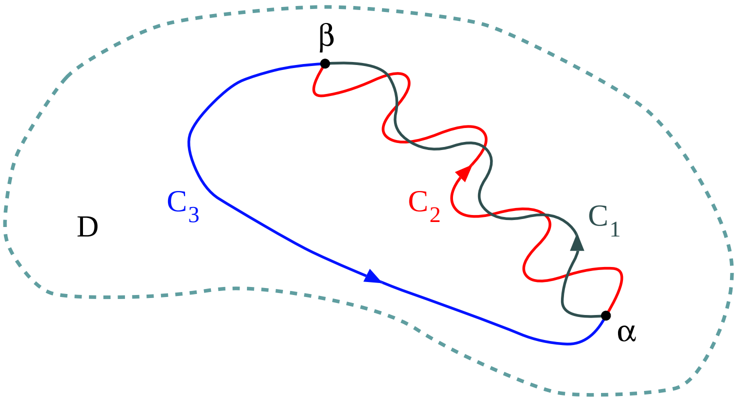
(note) As in Theorem 3, if we consider several simple closed contours in the positively oriented closed contour, then all are positively oriented, unless specifically stated otherwise.

5.4. **Principle of deformation of contours.**

Theorem 4. *(Principle of deformation of contours) Let $f(z)$ be holomorphic in a simply-connected domain. For a simple curve C from α to β, $\displaystyle\int_C f(z)dz$ is determined by only α and β, independent of the shape of C.*

Proof. For simple curves C_1, C_2 from α to β, consider a simple curve C_3 from β to α not intersecting C_1 nor C_2 as in the figure. Then $f(z)$ is holomorphic on and in the interior of the closed curve $C_1 + C_3$, and also holomorphic on and in the interior of $C_2 + C_3$. Therefore by Cauchy's integral theorem I,

$$\int_{C_1+C_3} f(z)dz = \int_{C_1} f(z)dz + \int_{C_3} f(z)dz = 0,$$
$$\int_{C_2+C_3} f(z)dz = \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 0. \tag{7}$$
$$\therefore \quad \int_{C_1} f(z)dz = -\int_{C_3} f(z)dz = \int_{C_2} f(z)dz. \quad \square$$



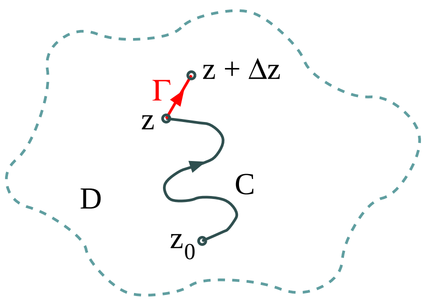
Similarly, the following corollary is proved. We also call it the principle of deformation of contours.

Theorem 4'. *Let $f(z)$ be continuous in a domain D. If $\int_\Gamma f(z)dz = 0$ for any simple closed curve Γ, then for a simple curve from α to β, $\int_C f(z)dz$ is determined by only α and β, independent of the shape of C.*

5.5. **Functions represented by the integrals.** Let D be a simply-connected domain, and let $f(z)$ be holomorphic in D. Fix a point z_0 and for a point z in D, take a curve C from z_0 to z in D. Since $\int_C f(\zeta)d\zeta$ is determined by only z, denote it by $\tilde{F}(z)$.

Theorem 5. *$\tilde{F}(z)$ is holomorphic in D and*

$$\tilde{F}'(z) = f(z). \tag{8}$$



Proof. Since $f(z)$ is holomorphic in a simply-connected domain, by the principle of deformation of contours, $\displaystyle\int_C f(\zeta)d\zeta$ is determined by only z and z_0, independent of the shape of C. Here, as z_0 is fixed, the integral is determined by only z. Hence we can set $\displaystyle\int_C f(\zeta)d\zeta = \tilde{F}(z)$. Take an arbitrary z in D, and for a change Δz of z, consider a line segment Γ from z to $z+\Delta z$, then we have

$$\begin{aligned} \tfrac{1}{\Delta z}\left[\tilde{F}(z+\Delta z)-\tilde{F}(z)\right] &= \tfrac{1}{\Delta z}\left(\int_{C+\Gamma} f(\zeta)d\zeta - \int_C f(\zeta)d\zeta\right) \\ &= \tfrac{1}{\Delta z}\int_\Gamma f(\zeta)d\zeta. \end{aligned} \tag{9}$$

Here taking the difference between this and $f(z)$, we have

$$\begin{aligned} \left|\tfrac{1}{\Delta z}\int_\Gamma f(\zeta)d\zeta - f(z)\right| &= \left|\tfrac{1}{\Delta z}\int_\Gamma f(\zeta)d\zeta - \tfrac{1}{\Delta z}\int_\Gamma f(z)d\zeta\right| \\ &= \left|\tfrac{1}{\Delta z}\int_\Gamma (f(\zeta)-f(z))d\zeta\right|. \end{aligned} \tag{10}$$

As $f(z)$ is holomorphic in D, it is of course continuous in D. Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that $|\Delta z| < \delta$ implies $|f(\zeta) - f(z)| < \epsilon$ $(\zeta \in \Gamma)$. Then by the estimation lemma,

$$\left|\frac{1}{\Delta z}\int_\Gamma (f(\zeta)-f(z))d\zeta\right| < \frac{1}{|\Delta z|}\cdot\epsilon\cdot|\Delta z| = \epsilon. \tag{11}$$

Accordingly, letting $\Delta z \to 0$,

$$\frac{1}{\Delta z}\left[\tilde{F}(z+\Delta z)-\tilde{F}(z)\right] = \frac{1}{\Delta z}\int_\Gamma f(\zeta)d\zeta \longrightarrow f(z) = \tilde{F}'(z). \tag{12}$$

Hence $\tilde{F}(z)$ is differentiable at z, and $\tilde{F}'(z) = f(z)$, where z is arbitrary point in D, and therefore $\tilde{F}(z)$ is holomorphic in D. □

5.6. **Primitive functions.** A function $F(z)$ which satisfies that $F'(z) = f(z)$ is called a primitive function or indefinite integral of a function $f(z)$. There are many primitive functions of $f(z)$, however, their differences are only constants, and they are expressed by one symbol $\displaystyle\int f(z)dz$.[2] (This is distinguished from a complex integral by the non-existence of a suffix to the integral symbol.) If $F(z)$ is one of primitive functions of $f(z)$, then

$$\int f(z)dz = F(z) + c \qquad (c \text{ is a complex integral constant.}) \tag{13}$$

[2]In a concrete calculation, $\int f(z)dz$ sometimes denotes one of primitive functions. See (16).

As we saw in Chapter 3, 3.2, 3.3, complex functions satisfy ordinary differential formulas. By the use of them, we have the following indefinite integral formulas.

$$
\begin{array}{lll}
\text{(i)} & \displaystyle\int (f(z)+g(z))dz = \int f(z)dz + \int g(z)dz & \\
\text{(ii)} & \displaystyle\int kf(z)dz = k\int f(z)dz & \\
\text{(iii)} & \displaystyle\int f(w)dw = \int f(g(z))g'(z)dz \quad (w=g(z)) & \text{(integration by substitution)} \\
\text{(iv)} & \displaystyle\int f'(z)g(z)dz = f(z)g(z) - \int f(z)g'(z)dz & \text{(integration by parts)}
\end{array}
\tag{14}
$$

Proof. (i): Let $F(z)$ and $G(z)$ be primitive functions of $f(z)$ and $g(z)$, respectively. The right-hand side is expressed as $F(z)+G(z)+c$. Differentiating it, we have $f(z)+g(z)$. Therefore the right-hand side is a primitive function of $f(z)+g(z)$, and it contains an integral constant. Hence we have (the right-hand side) $= \displaystyle\int (f(z)+g(z))dz$. (iii) The left-hand side is expressed as $F(z)+c$. Differentiating it by z, we have $f(w)g'(z) = f(g(z))g'(z)$. Therefore the left-hand side is a primitive function of $f(g(z))g'(z)$, and it contains an integral constant. Hence we have (the left-hand side) $= \displaystyle\int f(g(z))g'(z)dz$. (ii) and (iv) are proved similarly. □

5.7. **Primitive functions and complex integrals.** From Theorem 5, the following is derived.

Theorem 6. *Suppose $f(z)$ is holomorphic in a simply-connected domain D, and C is a simple curve in Dfrom z_0 to z. Take a primitive function $F(z)$ of $f(z)$, then*

$$\int_C f(\zeta)d\zeta = F(z) - F(z_0). \tag{15}$$

Proof. Letting $\displaystyle\int_C f(\zeta)d\zeta = \tilde{F}(z)$, by Theorem 5, $\tilde{F}'(z) = f(z)$. Meanwhile, for a primitive function $F(z)$ of $f(z)$, we have $F'(z) = f(z)$. Thus $\left(\tilde{F}(z) - F(z)\right)' = \tilde{F}'(z) - F'(z) = f(z) - f(z) = 0$. Therefore there exists a constant c and $\tilde{F}(z) - F(z) = c$. Here $\tilde{F}(z_0) - F(z_0) = 0 - F(z_0) = c$. Hence $\displaystyle\int_C f(\zeta)d\zeta = \tilde{F}(z) = F(z)+c = F(z)-F(z_0)$. □

(exercise01) Let $D : |z| < 3$ be a domain, and C be a simple curve from 0 to z in D. Calculate $\displaystyle\int_C \frac{\zeta-3}{\zeta+3}d\zeta$.

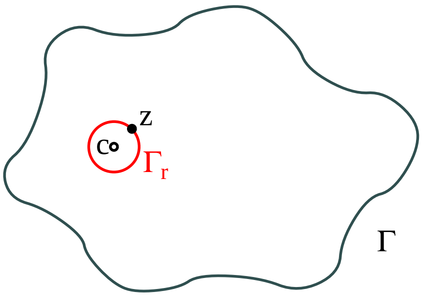
(answer) Since D is a simply-connected domain and $f(z) = \frac{z-3}{z+3}$ is holomorphic in D, we have $\int_C \frac{\zeta - 3}{\zeta + 3} d\zeta = F(z) - F(0)$ by a primitive function $F(z)$ of $f(z)$.

$$\begin{aligned} F(z) &= \int \frac{z-3}{z+3} dz = \int \left(1 - \frac{6}{z+3}\right) dz = z - 6\mathrm{Log}\,(z+3). \\ \therefore \quad \int_C \frac{\zeta-3}{\zeta+3} d\zeta &= \boxed{z - 6\mathrm{Log}\,(z+3) + 6\mathrm{Log}\,3}. \end{aligned} \tag{16}$$

5.8. **Cauchy's integral formula I.** Let Γ be a positively oriented simple closed curve, $f(z)$ be holomorphic function on and in the interior of Γ. Let c be a point in the interior of Γ.

Theorem 7. *(Cauchy's integral formula I) It holds that*

$$f(c) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-c} dz. \tag{17}$$



Proof. Denote by Γ_r the circle with the center c and radius r. Take sufficiently small $r > 0$ so that Γ_r could be contained in the interior of Γ. Then $\frac{f(z)}{z-c}$ is holomorphic on Γ and on Γ_r, and in the domain between them. Hence by Cauchy's integral theorem II, we have

$$\int_\Gamma \frac{f(z)}{z-c} dz = \int_{\Gamma_r} \frac{f(z)}{z-c} dz. \tag{18}$$

Here, we calculate the difference between this and $2\pi i f(c)$. Since $\displaystyle\int_{\Gamma_r} \frac{dz}{z-c} = 2\pi i$,

$$\begin{aligned} \int_\Gamma \frac{f(z)}{z-c} dz - 2\pi i f(c) &= \int_{\Gamma_r} \frac{f(z)}{z-c} dz - \int_{\Gamma_r} \frac{f(c)}{z-c} dz \\ &= \int_{\Gamma_r} \frac{f(z)-f(c)}{z-c} dz. \end{aligned} \tag{19}$$

As $f(z)$ is continuous in the interior of Γ, for any $\epsilon > 0$, taking sufficiently small $r > 0$, we have $|f(z) - f(c)| < \epsilon$ on Γ_r. Then by the ML inequality,

$$\left| \int_{\Gamma_r} \frac{f(z)-f(c)}{z-c} dz \right| < \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon. \tag{20}$$

That is,

$$\left|\int_\Gamma \frac{f(z)}{z-c}dz - 2\pi i f(c)\right| < 2\pi\epsilon. \tag{21}$$

Here, as ϵ is an arbitrary positive number, it must hold that

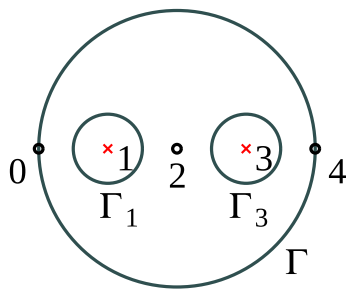
$$\int_\Gamma \frac{f(z)}{z-c}dz = 2\pi i f(c). \tag{22}$$

Dividing both sides by $2\pi i$, we have (17). □

This theorem says that if $f(z)$ is holomorphic on and in the interior of Γ, then all values of $f(z)$ in the interior of Γ is completely determined by its values on Γ, which is a quite amazing phenomenon.

(exercise02)

For the positively oriented circle $\Gamma : |z-2| = 2$, calculate $\displaystyle\int_\Gamma \frac{z-2}{(z-1)(z-3)}dz$.



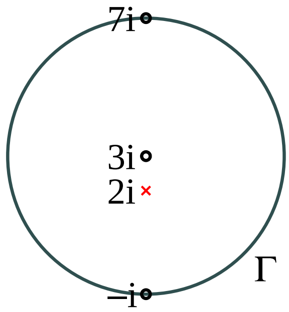
(answer) Take Γ_1 and Γ_3 as in the figure. By Cauchy's integral theorem II,

$$\int_\Gamma \frac{z-2}{(z-1)(z-3)}dz = \int_{\Gamma_1} \frac{z-2}{(z-1)(z-3)}dz + \int_{\Gamma_3} \frac{z-2}{(z-1)(z-3)}dz. \tag{23}$$

Here, let $\frac{z-2}{z-3} = f(z)$ and $\frac{z-2}{z-1} = g(z)$, then $f(z)$ is holomorphic on and in the interior of Γ_1, and $g(z)$ is holomorphic on and in the interior of Γ_3. Hence by Cauchy's integral formula I,

$$(23) = \int_{\Gamma_1} \frac{f(z)}{z-1}dz + \int_{\Gamma_3} \frac{g(z)}{z-3}dz = 2\pi i f(1) + 2\pi i g(3) = \boxed{2\pi i}. \tag{24}$$

(exercise03) For the positively oriented circle $\Gamma : |z-3i| = 4$, calculate $\displaystyle\int_\Gamma \frac{e^{\pi z}}{z^2+4}dz$.



(answer) Let $\frac{e^{\pi z}}{z^2+4} = \frac{e^{\pi z}}{(z+2i)(z-2i)} = f(z)$. The singularity of $f(z)$ in the interior of Γ is only $z = 2i$. Hence let $\frac{e^{\pi z}}{z+2i} = g(z)$, then $g(z)$ is holomorphic on and in the interior of Γ. Therefore by Cauchy's integral formula I,

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{g(z)}{z-2i}dz = 2\pi i g(2i) = 2\pi i \frac{e^{2\pi i}}{2i+2i} = \boxed{\frac{\pi}{2}}. \tag{25}$$

CHAPTER 6

CAUCHY'S INTEGRAL FORMULA II AND ITS APPLICATIONS

⋆ **6** ⋆

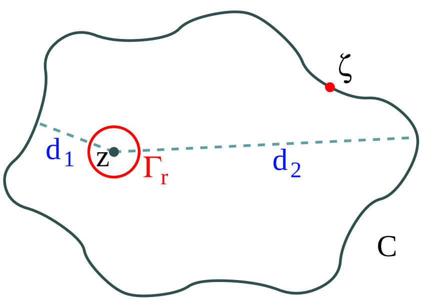
KEYWORDS: CAUCHY'S INTEGRAL FORMULA II, MORERA'S THEOREM, MAXIMUM MODULUS PRINCIPLE, CAUCHY'S EVALUATION FORMULA, LIOUVILLE'S THEOREM, FUNDAMENTAL THEOREM OF ALGEBRA

6.1. **Cauchy's integral formula II.** The following is a generalization of Cauchy's integral formula I, which is contained as the case of $n = 0$.

Theorem 1. *(Cauchy's integral formula II) Let D be a domain and $f(z)$ be holomorphic in D. Then $f(z)$ is differentiable for arbitrary times, and the n-th derivative of $f(z)$ is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta. \tag{1}$$

Here, C is a positively oriented simple closed curve such that C and its interior is contained in D, and z is a point in the interior of C.



Proof. We prove this theorem by induction on n. Let $f(z)$ be holomorphic in D. Let C and z be as above. Then by Cauchy's integral formula I,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{2}$$

This formula shows that (1) is valid for $n = 0$.

Suppose (1) holds for $n-1$, then we have

$$\begin{aligned}
&\tfrac{2\pi i}{(n-1)!}\cdot\tfrac{1}{h}\left[f^{(n-1)}(z+h)-f^{(n-1)}(z)\right]=\frac{1}{h}\left(\int_C\frac{f(\zeta)}{(\zeta-(z+h))^n}d\zeta-\int_C\frac{f(\zeta)}{(\zeta-z)^n}d\zeta\right)\\
&=\frac{1}{h}\int_C f(\zeta)\frac{(\zeta-z)^n-(\zeta-z-h)^n}{(\zeta-z-h)^n(\zeta-z)^n}d\zeta\\
&=\frac{1}{h}\int_C f(\zeta)\frac{(\zeta-z)-(\zeta-z-h)}{(\zeta-z-h)^n(\zeta-z)^n}\left[\textstyle\sum_{k=0}^{n-1}(\zeta-z)^k(\zeta-z-h)^{n-k-1}\right]d\zeta\\
&=\int_C\frac{f(\zeta)}{(\zeta-z-h)^n(\zeta-z)^n}\left[\textstyle\sum_{k=0}^{n-1}(\zeta-z)^k(\zeta-z-h)^{n-k-1}\right]d\zeta.
\end{aligned}\tag{3}$$

Here, we evaluate the difference between this and $n\displaystyle\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$. For simplicity, write $\zeta-z=s$.

$$\begin{aligned}
&\int_C\frac{f(\zeta)}{(s-h)^ns^n}\left[\textstyle\sum_{k=0}^{n-1}s^k(s-h)^{n-k-1}\right]d\zeta-n\int_C\frac{f(\zeta)}{s^{n+1}}d\zeta\\
&=\int_C\frac{f(\zeta)}{(s-h)^ns^{n+1}}\left[s\textstyle\sum_{k=0}^{n-1}s^k(s-h)^{n-k-1}-n(s-h)^n\right]d\zeta\\
&=\int_C\frac{f(\zeta)}{(s-h)^ns^{n+1}}\left[\textstyle\sum_{k=0}^{n-1}\left(s^{k+1}(s-h)^{n-k-1}-(s-h)^n\right)\right]d\zeta\\
&=\int_C\frac{f(\zeta)}{(s-h)^ns^{n+1}}\left[\textstyle\sum_{k=0}^{n-1}(s-h)^{n-k-1}\left(s^{k+1}-(s-h)^{k+1}\right)\right]d\zeta\\
&=\int_C\frac{f(\zeta)}{(s-h)^ns^{n+1}}\left[\textstyle\sum_{k=0}^{n-1}(s-h)^{n-k-1}h\sum_{l=0}^{k}s^l(s-h)^{k-l}\right]d\zeta\\
&=h\int_C\frac{f(\zeta)}{(s-h)^ns^{n+1}}\left[\textstyle\sum_{k=0}^{n-1}\sum_{l=0}^{k}s^l(s-h)^{n-l-1}\right]d\zeta
\end{aligned}\tag{4}$$

Take sufficiently small $r>0$, so that the circle Γ_r with the center z and radius r, and the interior of Γ_r are contained in D. Then if $|h|<r$, then $z+h$ is contained in the interior of Γ_r. Let d_1 be the distance between z and C, d_2 be the maximum value of $|s|$ $(\zeta\in C)$, then we have $d_1-r\le|s|,|s-h|\le d_2+r$. Let M be the maximum value of $|f(\zeta)|$ on C, and L be the length of C. By the use of these values, evaluating the modulus of the last expression of (4),

$$|\cdots|\le|h|\cdot\frac{M}{(d_1-r)^{2n+1}}\cdot\frac{n(n+1)}{2}\cdot(d_2+r)^{n-1}\cdot L.\tag{5}$$

Hence letting $|h|\to 0$, we have

$$\begin{aligned}
&\tfrac{2\pi i}{(n-1)!}\cdot\tfrac{1}{h}\left[f^{(n-1)}(z+h)-f^{(n-1)}(z)\right]\longrightarrow n\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.\\
&\therefore\quad \tfrac{1}{h}\left[f^{(n-1)}(z+h)-f^{(n-1)}(z)\right]\longrightarrow\tfrac{n!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=f^{(n)}(z).
\end{aligned}\tag{6}$$

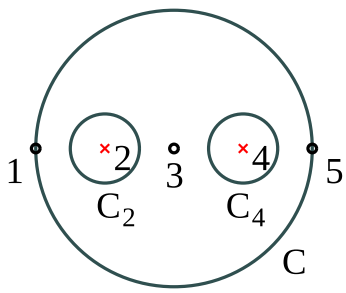
Therefore (1) holds for n, and the induction is completed. □

As we saw here, a function differentiable once in some domain, is automatically differentiable arbitrary times there, which is completely different from the real function case, very mysterious property.
(note) The formula (1) is derived by partially differentiating the integrand on the right-hand side of (2) n times by z. However, this method is not verified at this stage, and so we gave the above long proof. For your reference, we introduce a theorem which assures the properness of the method. A proof of this theorem is given in Chapter 12.

Theorem 2. *Let $g(z,\zeta)$ a complex function of complex variables z and ζ, where z runs over a domain D and ζ runs over a curve C. Suppose that $g(z,\zeta)$ is continuous, and for every fixed $\zeta \in C$, $g(z,\zeta)$ is holomorphic in D as a function of z. Then $f(z) = \int_C g(z,\zeta)d\zeta$ is holomorphic in D and*

$$f'(z) = \int_C \frac{\partial}{\partial z} g(z,\zeta)d\zeta. \tag{7}$$

(exercise01)
For the positively oriented circle $C : |z-3| = 2$, calculate $\displaystyle\int_C \frac{\operatorname{Log} z}{(z-2)^2(z-4)}dz$.



(answer) Take C_2 and C_4 as in the figure. By Cauchy's integral theorem II,

$$\int_C \frac{\operatorname{Log} z}{(z-2)^2(z-4)}dz = \int_{C_2} \frac{\operatorname{Log} z}{(z-2)^2(z-4)}dz + \int_{C_4} \frac{\operatorname{Log} z}{(z-2)^2(z-4)}dz. \tag{8}$$

Here, let $\frac{\operatorname{Log} z}{z-4} = f(z)$ and $\frac{\operatorname{Log} z}{(z-2)^2} = g(z)$, then $f(z)$ is holomorphic on and in the interior of C_2, and $g(z)$ is holomorphic on and in the interior of C_4. Hence by Cauchy's integral formula I and II,

$$\begin{aligned}(8) \quad &= \int_{C_2} \frac{f(z)}{(z-2)^2}dz + \int_{C_4} \frac{g(z)}{z-4}dz = 2\pi i f'(2) + 2\pi i g(4) \\ &= 2\pi i \left[\frac{\frac{1}{z}(z-4) - \operatorname{Log} z}{(z-4)^2}\right]_{z=2} + 2\pi i \frac{\operatorname{Log} 4}{4} = \boxed{\frac{\operatorname{Log} 2 - 1}{2}\pi i}.\end{aligned} \tag{9}$$

Hereinafter, we study several applications of Cauchy's integral formula.

6.2. **Morera's theorem.**

Theorem 3. *(Morera's theorem) Let $f(z)$ be continuous in a domain D. Suppose $\int_\Gamma f(z)dz = 0$ for every simple closed curve Γ, then $f(z)$ is holomorphic in D.*

Proof. Fix a point z_0 in D, and for a point z in D, take a curve C in D from z_0 to z. Then by the principle of deformation of contours, $\int_C f(\zeta)d\zeta$ is determined by only z, and so we denote it by $\tilde{F}(z)$. As we saw in Chapter 5, 5.5, $\tilde{F}(z)$ is holomorphic in D, and $\tilde{F}'(z) = f(z)$. However, by Section 6.1, every holomorphic function is differentiable for arbitrary times, and therefore $\tilde{F}'(z) = f(z)$ is also differentiable in D, that is, $f(z)$ is holomorphic in D. □

This theorem holds unless D is simply connected. If D is simply connected, this is considered as the inverse of Cauchy's integral theorem I.

6.3. **Maximum modulus principle.**

Theorem 4. *(Maximum modulus principle) Let Γ be a simple closed curve, and E be the set of all points on and in the interior of Γ. Let $f(z)$ be a holomorphic function on E. Then $|f(z)|$ has global maxima at points on Γ.*

Proof. Let c be an arbitrary point in the interior of Γ. Let M be the maximum value of $|f(z)|$ on Γ, L be the length of Γ, d be the distance between c and Γ. Since $(f(z))^n$ $(n = 1, 2, \dots)$ is holomorphic on E, we have, by Cauchy's integral formula I and the estimation lemma,

$$\begin{aligned} &|f(c)|^n = |(f(c))^n| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{(f(z))^n}{z-c} dz \right| \leq \frac{1}{2\pi} \cdot \frac{M^n}{d} \cdot L. \\ &\therefore \quad |f(c)| \leq M \frac{L^{1/n}}{(2\pi d)^{1/n}} \longrightarrow M. \quad (n \to \infty) \quad \square \end{aligned} \tag{10}$$

(note) $|f(z)|$ has no global maxima at the point in the interior of Γ, unless $f(z)$ is a constant.
(exercise02) Determine the global maximum of $f(z) = e^z$ on $|z - i| \leq 3$.

6.4. Cauchy's evaluation formula.

Theorem 5. *(Cauchy's evaluation formula) Let Γ_r be the circle with the center c and radius $r > 0$, and $f(z)$ be holomorphic on and in the interior of Γ_r. Let M be the maximum value of $|f(z)|$ on Γ_r. Then we have*

$$\left|f^{(n)}(c)\right| \leq \frac{n!M}{r^n}. \tag{11}$$

Proof. By Cauchy's integral formula II and the estimation lemma,

$$\left|f^{(n)}(c)\right| = \left|\frac{n!}{2\pi i}\int_{\Gamma_r}\frac{f(z)}{(z-c)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{n!M}{r^n}. \quad \square \tag{12}$$

As an application of this theorem, we have he following Liouville's theorem, and a further application of it, we have the fundamental theorem of algebra.

6.5. Liouville's theorem.

Theorem 6. *(Liouville's theorem) Every entire function bounded in the whole complex plane is necessarily a constant.*

Proof. Let $f(z)$ be an entire function bounded in the whole complex plane. Take an arbitrary point c, and let Γ_r be the circle with the center c and radius r. Since $f(z)$ is bounded, there exists a constant $\tilde{M}$ such that $|f(z)| \leq \tilde{M}$ for every z, and letting the maximum value of $|f(z)|$ on Γ_r be M, we have $M \leq \tilde{M}$ (#). Here, applying Cauchy's evaluation formula for $n = 1$, it holds that

$$|f'(c)| \leq \frac{M}{r}. \tag{13}$$

Combining this with (#), we have

$$|f'(c)| \leq \frac{\tilde{M}}{r}. \tag{14}$$

Then letting $r \to \infty$, we have $|f'(c)| = 0$. As c is arbitrary, $f'(z) = 0$ in the whole complex plane. Hence we have $f(z) = \text{const}$. $\square$

6.6. **Fundamental theorem of algebra.**

Theorem 7. *(Fundamental theorem of algebra) Every algebraic equation with complex coefficients has at least one solution in the complex plane.*

Proof. Take an arbitrary polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ ($n \geq 1$, $a_0 \neq 0$) with complex coefficients, and consider a function:

$$g(z) = \frac{1}{f(z)} = \frac{1}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}. \tag{15}$$

Suppose the algebraic equation $f(z) = 0$ has never complex solutions. Then $g(z)$ is holomorphic in the whole complex plane, say, an entire function. Letting $z \to \infty$, we have

$$\begin{aligned} |g(z)| &= \frac{1}{|a_0 z^n + a_1 z^{n-1} + \cdots + a_n|} \\ &= \frac{1}{|z|^n} \cdot \frac{1}{\left|a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}\right|} \longrightarrow 0. \end{aligned} \tag{16}$$

Hence for sufficiently large R, $|z| > R$ implies $|g(z)| < 1$. In addition, $g(z)$ is holomorphic and so it is continuous, thus $|g(z)|$ is also continuous. Therefore $|g(z)|$ has the maximum value M on $|z| \leq R$. Consequently, in the whole complex plane,

$$|g(z)| \leq \max(1, M), \tag{17}$$

that is, $g(z)$ is bounded in the whole complex plane. However, $f(z)$ is an entire function, thus by Liouville's theorem, $f(z)$ is a constant, which is clearly a contradiction. Accordingly, $f(z) = 0$ should have a complex solution. □

Once we accept this theorem, for a polynomial $f(z)$, choosing a complex solution α_1, by the factor theorem, we have

$$f(z) = (z - \alpha_1) f_1(z). \tag{18}$$

Similarly, choosing a complex solution α_2 of $f_1(z)$, $f(z) = (z - \alpha_1)(z - \alpha_2) f_2(z)$. Repeating this process gives

$$f(z) = a_0 (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n). \tag{19}$$

That is, $f(z)$ is completely factored into linear factors with complex coefficients.