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LINEAR ALGEBRA EXERCISES

1. Given the three points $P(1,-2,3)$, $Q(4,-4,1)$, $R(1,0,2)$ in the 3-dimensional space, answer the following.

(1) Determine S satisfying $\overrightarrow{PQ} = \overrightarrow{RS}$.
(2) Determine the center of gravity G of $\triangle PQR$. (You may use the formula for the center of gravity.)
(3) Letting the center of gravity of $\triangle QRS$ be H, determine $\overrightarrow{GH}$.

1.1* Given a tetrahedron $O - ABC$ in the 3-dimensional space. If we draw a line segment between each vertex and the center of gravity of the opposite face, then show that those four segments meets at one point. (This point is called the center of gravity of the tetrahedron.)

2. Let k be a real number and $\mathtt{a}, \mathtt{b}, \mathtt{c}$ be 3-dimensional vectors. Show the following.

(1) $|(\mathtt{a}, \mathtt{b})| \le ||\mathtt{a}|| \cdot ||\mathtt{b}||$ (2) $||\mathtt{a}+\mathtt{b}||^2 - ||\mathtt{a}-\mathtt{b}||^2 = 4(\mathtt{a}, \mathtt{b})$
(3) $||k\mathtt{a}|| = |k|\, ||\mathtt{a}||$ (4) $||\mathtt{a}+\mathtt{b}|| \le ||\mathtt{a}|| + ||\mathtt{b}||$
(5) $||\mathtt{a}+\mathtt{b}+\mathtt{c}||^2 - ||\mathtt{a}||^2 - ||\mathtt{b}||^2 - ||\mathtt{c}||^2 = 2[(\mathtt{a}, \mathtt{b}) + (\mathtt{b}, \mathtt{c}) + (\mathtt{c}, \mathtt{a})]$

3. Are the following vectors linearly independent?

$$(1)\ \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \qquad (2)\ \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

4. Letting $\mathtt{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathtt{b} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \mathtt{c} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, express the following vectors by linear combinations of $\mathtt{a}, \mathtt{b}, \mathtt{c}$.

$$(1)\ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad (2)\ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad (3)\ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \qquad (4)\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

4.1 Letting $\mathtt{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathtt{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathtt{c} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, express $\mathtt{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ by a linear combination of $\mathtt{a}, \mathtt{b}, \mathtt{c}$.

5. Determine the vector equation of the following line (1) in the plane, and lines (2),(3) in the 3-dimensional space.

(1) $9x + 4y = 5$ (2) $-x - 3 = 2y = z$ (3) $4x + 3 = 6y + 5 = -2z + 1$

6. Determine the vector equation of a line $\begin{cases} x + y - 2z = 5 \\ x - y + 3z = 1 \end{cases}$.

7. Let θ be the dihedral angle of two planes $-x+2y-5z = 8$ and $-2x-y+3z = 19$, then determine $\sin\theta$. (The dihedral angle of two planes is the angle of their normal vectors. If the direction of the normal vector of one plane is changed, we have another value of θ, but $\sin\theta$ is uniquely determined.)

8. Determine the equations of the following planes and lines expressed by vector equations.

(1) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + u\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ (2) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + u\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

(3) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ (4) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix}$

(5) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + u\begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

9. Rewrite the following equations of planes into vector equations.

(1) $x + 2y - z = 3$. (2) $3x + 2y + z = 0$.
(3) $5x - 8z = 25$.

10. Calculate the following.

(1) $\begin{pmatrix} 5 & 15 & 10 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & -1 \\ -2 & 2 \end{pmatrix}$ (2) $AB = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

(3) BA

11.* The area S of a parallelogram spanned by two 3-dimensional vectors $\mathsf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathsf{a}' = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$ is expressed as $S = \sqrt{S_1^2 + S_2^2 + S_3^2}$ by the areas S_1, S_2, S_3 of the orthogonal projections to xy, yz, zx planes, respectively. Show this fact.

12. For the following linear transformations T of V^3, determine the matrices A satisfying $T\mathtt{x} = A\mathtt{x}$ ($\mathtt{x} \in V^3$). Also, for (1) and (3), calculate A^n.

$$(1)\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

$$(2)\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x-2y+3z \\ 3x+5y-2z \\ -2x+3y+5z \end{pmatrix}$$

$$(3)\ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y-z \\ -x+2y-z \\ -x-y+2z \end{pmatrix}$$

13. For the following linear transformations T, S of V^3 answer the following: (1) Determine the matrix A satisfying $ST = T_A$. (2) Determine the matrix B satisfying $TS = T_B$.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x-2y-3z \\ x+2y+2z \\ 2x-y-3z \end{pmatrix}, \qquad S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-2y+z \\ x-2y+2z \\ 2x+2y-z \end{pmatrix}$$

14. Consider the 3×3 matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$. For the following line l or plane S, determine the equation of the images l' and S' of l and S, respectively, under T_A. Here, d in (5) is a real constant.

(1) $\dfrac{x}{3} = -y-2 = \dfrac{z+1}{2}$ (2) $-\dfrac{x+2}{10} = \dfrac{y-3}{3} = \dfrac{z-3}{12}$

(3) $3x+y-4z=2$ (4) $2x-3y+3z=3$

(5) $y-z=d$

15. Consider the 3×3 matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$. For the following line l or plane S, determine the equation of the images l' and S' of l and S, respectively, under T_A.

(1) $2x = -y = 3z$ (2) $5x+2y+6z=0$

16. Consider the 3×3 matrix $A = k\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Suppose the image of the plane $S_d : x+y+z=d$ under T_A is S_d itself for every real constant d. Determine the value of k.

17. (1) Let A be a matrix which determines the linear transformation mapping every point in the x-y plane to the point symmetric to it with respect to the line $l : y = 2x$. Determine A.

(2) Solve the similar problem for the line $l : y = mx$.

18. (1) Let A, B be square matrices of order 2 or 3, then show that $|AB| = |A||B|$.

(2) Let A be a square matrix of order 2 or 3, then show that $|A| = |{}^tA|$.

19. Two matrices A and B are called commutative if it holds that $AB = BA$. Determine all matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ commutative with the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

20. Prove the following equalities.

$$(1) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -(x-y)(x-z)(y-z)$$

$$(2) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$$

Here, $\omega = \frac{-1+\sqrt{3}i}{2}$. (One of the primitive third root of unity)

21. Let $\mathtt{x} = (1, 2, \ldots, n)$ and $\mathtt{y} = (1, t, \ldots, t^{n-1})$. Calculate the following.

(1) $\mathtt{x}^t\mathtt{y}$ (2) ${}^t\mathtt{x}\mathtt{y}$ (3) $\mathtt{x}^t\mathtt{x}$ (4) ${}^t\mathtt{y}\mathtt{y}$ (5) ${}^t\mathtt{x}\mathtt{y}^t({}^t\mathtt{x}\mathtt{y})$

21.1 For $\mathtt{a} = (a_1, \ldots, a_n)$, $\mathtt{b} = (b_1, \ldots, b_n)$, calculate $({}^t\mathtt{a}\mathtt{b})^n$.

22. Using block matrices, calculate the following. Here, A is a nonsingular matrix of order 2, and E is the identity matrix of order 2.

$$(1) \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 \qquad (2) \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$(3) \begin{pmatrix} A & -E \\ E & A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & E - A^{-1} & -E \\ -E & E + A & -A \end{pmatrix}$$

23.* Let A be an $l \times m$ matrix and B be an $m \times n$ matrix, then prove that ${}^t(AB) = {}^tB{}^tA$.

24. Let $A = \begin{pmatrix} P & Q \\ O & S \end{pmatrix}$ be a symmetrically partitioned matrix, where P is $r \times r$, and S is $s \times s$. Then prove that a necessary and sufficient condition for A to be nonsingular is that P and S is nonsingular. (You may use the theorem that for a square matrix M, if $MX = E$ or $XM = E$ holds, then M is nonsingular.)

25.* Let $\Theta_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Using block matrices, calculate the following.

$$\begin{pmatrix} (\cos\alpha)\ \Theta_\theta & (-\sin\alpha)\ \Theta_\theta \\ (\sin\alpha)\ \Theta_\theta & (\cos\alpha)\ \Theta_\theta \end{pmatrix} \begin{pmatrix} (\cos\beta)\ \Theta_\phi & (-\sin\beta)\ \Theta_\phi \\ (\sin\beta)\ \Theta_\phi & (\cos\beta)\ \Theta_\phi \end{pmatrix}$$

26. Using elementary operations, transform the following matrices into the form $\begin{pmatrix} E_r & O \\ O & O \end{pmatrix}$, and determine the ranks of them. Here, x is a real number.

(1) $\begin{pmatrix} 6 & -1 & -1 \\ 1 & 1 & 1 \\ 8 & 1 & 1 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 2 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 2 & 3 & 4 & 5 \\ 9 & 8 & 7 & 6 \end{pmatrix}$ (3) $\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & 5 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 5 & 5 \end{pmatrix}$

(4) $\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$ (5) $\begin{pmatrix} 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

(6) $\begin{pmatrix} x & & O & 1 \\ & x & & 1 \\ & & \ddots & \vdots \\ O & & & \\ 1 & 1 & \dots & x \end{pmatrix}$ (of order n, $n \geq 2$)

27. Determine the inverse matrices of the following matrices. Here, A and B are nonsingular.

(1) $\begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} -4 & 5 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ (3) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ -3 & -2 & 0 & 0 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$

(4) $\begin{pmatrix} 2 & 1 & 1 & 0 \\ -3 & -2 & 0 & 1 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$ (5) $\begin{pmatrix} 3 & -5 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -2 \end{pmatrix}$ (6) $\begin{pmatrix} E & A \\ B & O \end{pmatrix}$

28. Applying elementary row operations on $(A \;\; E)$, determine the inverse matrix of A if it is nonsingular. Here, a, b are real numbers satisfying that $|a| \neq |b|$.

(1) $\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$ (2) $\begin{pmatrix} 0 & 0 & a & -1 & 1 & 0 & 0 & 0 \\ 0 & a & -1 & 0 & 0 & 1 & 0 & 0 \\ a & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(3) $\begin{pmatrix} 7 & -8 & 8 & 1 & 0 & 0 \\ -4 & 5 & -6 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{pmatrix}$ (4)* $\begin{pmatrix} a & 0 & 0 & b & 1 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 1 & 0 & 0 \\ 0 & b & a & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & a & 0 & 0 & 0 & 1 \end{pmatrix}$

29. For nonsingular $n \times n$ matrices A, B, show that $(AB)^{-1} = B^{-1}A^{-1}$.

30. For a nonsingular $n \times n$ matrix A, show that $\left({}^tA\right)^{-1} = {}^t\left(A^{-1}\right)$.

31.* Let $n \geq 2$. Calculate the rank of the $n \times n$ matrix $\begin{pmatrix} 1 & x & x & \dots & x \\ x & 1 & x & \dots & x \\ x & x & 1 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \dots & 1 \end{pmatrix}$ as follows.

(1) Add all columns except a specified column (e.g. the n-th column) to the specified column.

(2) After dividing into cases, add a scalar multiple of the n-th column to each column, and cope with it.

32. Solve the following linear equations, where x, y, z, w are variables and a is a constant. (Consider similarly to the multi-equation case.)

$$\begin{array}{ll} (1) & x + ay - az - aw = a \\ (2) & ax + 0y - 5z - w = 4 \\ (3) & 3x + 2y = z + 4w \end{array}$$

33. Solve the following systems of linear equations, where a, b, c are constants.

$$(1) \left\{ \begin{array}{rrrrcr} x & +y & +z & & = & 4 \\ & y & & -w & = & 5 \\ x & +y & & +w & = & 6 \end{array} \right. \quad (2) \left\{ \begin{array}{rrrrcr} 3x & +2y & +z & -w & = & 4 \\ 2x & +y & & -w & = & -2 \\ 4x & +3y & +2z & -w & = & 10 \end{array} \right.$$

$$(3) \left\{ \begin{array}{rrrcr} & -y & +z & = & -2 \\ 5x & -y & -4z & = & 3 \\ x & & -z & = & 1 \end{array} \right. \quad (4) \left\{ \begin{array}{rrrrcr} -3x & +y & -z & +w & = & -8 \\ -2x & -2y & -2z & +3w & = & -12 \\ 3x & +y & +z & -2w & = & 12 \end{array} \right.$$

$$(5) \left\{ \begin{array}{rrrcr} u & +v & +w & = & a \\ w & +x & +y & = & b \\ y & +z & +u & = & c \end{array} \right. \quad (6) \left\{ \begin{array}{rrrrcr} & y & +z & +2w & = & 0 \\ 2x & +4y & +6z & +8w & = & 0 \\ 2x & & +2z & & = & 0 \end{array} \right.$$

$$(7) \left\{ \begin{array}{rrrrcr} & & z & +2w & = & 0 \\ x & +2y & & & = & 0 \end{array} \right. \quad (8)^* \left\{ \begin{array}{rrrrcr} 2x & +3y & -5z & -2w & = & 9 \\ -2x & +2y & +3z & -5w & = & -18 \\ -5x & -2y & +2z & +3w & = & -7 \\ 3x & -5y & -2z & +2w & = & 4 \end{array} \right.$$

34. Determine a necessary and sufficient condition for the following systems of linear equations to have a solution, and solve them under the condition. Here, for (2), suppose $abc \neq 0$.

$$(1) \left\{ \begin{array}{rrrcr} 2x & +5y & +6z & = & a \\ x & +2y & +z & = & b \\ 3x & +5y & -z & = & c \end{array} \right. \quad (2) \left\{ \begin{array}{rrrcr} ax & -by & & = & c \\ & by & -cz & = & a \\ -ax & & +cz & = & b \end{array} \right.$$

35. Solve the following systems of linear equations, where a, b, c are constants.

$$(1) \left\{ \begin{array}{rrrrcr} & 2y & +z & +w & = & a \\ x & & -z & -2w & = & b \\ x & +2y & & -w & = & 2 \end{array} \right. \quad (2) \left\{ \begin{array}{rrrrcr} 2x & +2y & +2z & +w & = & 2 \\ 3x & +2y & +2z & +w & = & 1 \\ 2x & +y & +z & +w & = & 4 \end{array} \right.$$

$$(3)^* \left\{ \begin{array}{rrrrcr} x & +ay & +z & -w & = & 3 \\ x & -y & -z & +bw & = & 5 \end{array} \right. \quad (4)^* \left\{ \begin{array}{rrrcr} bu & +v & +cw & = & a \\ cw & +x & +ay & = & b \\ ay & +z & +bu & = & c \end{array} \right.$$

36. Let A be a 3×3 matrix, $\mathtt{c}$ be a column vector with 3 entries, and $\mathtt{x} = {}^t(x, y, z)$. Hitomi tried to solve the equation $A\mathtt{x} = \mathtt{c}$, and applied elementary operations to the extended coefficient matrix $(A\ \mathtt{c})$, then she multiplied the second column by 2, by mistake. She did not interchange the columns. After all, she has a solution $\mathtt{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$. Find the proper solution.

37. Calculate the following determinants.

$$(1)\ \begin{vmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{vmatrix} \quad (2)\ \begin{vmatrix} 2 & 5 & 2 & 2 \\ 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 5 \\ 5 & 5 & 5 & 5 \end{vmatrix} \quad (3)\ \begin{vmatrix} a & 0 & 0 & e \\ 0 & b & f & 0 \\ 0 & g & c & 0 \\ h & 0 & 0 & d \end{vmatrix} \quad (4)\ \begin{vmatrix} a & b & & O \\ 0 & a & b & \\ 0 & 0 & a & b \\ b & 0 & 0 & a \end{vmatrix}$$

$$(5)\ \begin{vmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} \quad (6)\ \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \quad (7)\ \begin{vmatrix} 1 & 14 & 15 & 4 \\ 12 & 7 & 6 & 9 \\ 8 & 11 & 10 & 5 \\ 13 & 2 & 3 & 16 \end{vmatrix}$$

38. Determine all the value of the complex number c such that the following matrices are singular.

$$(1)\ \begin{pmatrix} c+1 & -2 \\ -1 & c \end{pmatrix} \quad (2)\ \begin{pmatrix} c & -1 & 1 \\ 0 & c & 2 \\ c & 0 & 1 \end{pmatrix} \quad (3)\ \begin{pmatrix} c & c & c+9 \\ c+1 & c+2 & c+12 \\ 0 & 9 & c \end{pmatrix}$$

$$(4)\ \begin{pmatrix} c & 1 & 2 & 3 \\ 1 & c+2 & 3 & 0 \\ 2 & 3 & c & 1 \\ 3 & 0 & 1 & c+2 \end{pmatrix} \quad (5)\ \begin{pmatrix} c & 1 & 2 & 3 \\ -1 & c & 1 & 2 \\ -2 & -1 & c & 1 \\ -3 & -2 & -1 & c \end{pmatrix}$$

39. Decompose the following permutations into transpositions, and determine the signs of the permutations.

$$(1)\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad (2)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad (3)\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

40. Calculate the following determinants. (For (1), (2), decompose into factors. For (5) you may assume that $t = 1$.)

$$(1)^*\ \begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix} \quad (2)\ \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} \quad (3)^*\ \begin{vmatrix} x & 1 & 3 & \dots & 2n-1 \\ 1 & x & 3 & \dots & 2n-1 \\ 1 & 3 & x & \dots & 2n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 5 & \dots & x \end{vmatrix}$$

$$(4)^*\ \begin{vmatrix} 15 & 8 & 1 & 24 & 17 \\ 16 & 14 & 7 & 5 & 23 \\ 22 & 20 & 13 & 6 & 4 \\ 3 & 21 & 19 & 12 & 10 \\ 9 & 2 & 25 & 18 & 11 \end{vmatrix} \quad (5)^{**}\ \underbrace{\begin{vmatrix} t^{-1}+tx^2 & x & & & O \\ x & t^{-1}+tx^2 & x & & \\ & \ddots & \ddots & \ddots & \\ & & x & t^{-1}+tx^2 & x \\ O & & & x & t^{-1}+tx^2 \end{vmatrix}}_{n}$$

41. Show that the determinant of a skew-symmetric matrix (${}^tA = -A$) of odd order is equal to 0.

42. Given three points (x_i, y_i, z_i) $(i = 1, 2, 3)$ in the 3-dimensional space, not on the same line, express the equation of the plane containing the three points by a determinant.

43. Calculate the following. Here, $n = 1, 2, 3, 4$ for (3).

$$(1)\ \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}^{-1} \quad (2)\ \begin{pmatrix} 1 & -2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{pmatrix}^{-1} \quad (3)\ \underbrace{\begin{vmatrix} k & 1 & & & O \\ 1 & k & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & k & 1 \\ O & & & 1 & k \end{vmatrix}}_{n}$$

$$(4)^*\ \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_1^2 & x_2^2 & \dots & x_{n-1}^2 & x_n^2 \\ \multicolumn{5}{c}{\dotfill} \\ x_1^{n-1} & x_2^{n-1} & \dots & x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix}$$

44. Using the inverse matrix formula $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}\,(A)$, calculate the inverses of the following matrices.

$$(1)\ \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \\ 1 & 4 & -2 \end{pmatrix} \qquad (2)\ \begin{pmatrix} 3 & 5 & a \\ 2 & 3 & 0 \\ 2 & 3 & 2 \end{pmatrix}$$

45. Solve the following systems of linear equations using Cramer's rule. Here, for (2), suppose that $a^3 + b^3 + c^3 \neq 3abc$.

$$(1)\ \left\{ \begin{array}{rrrcr} 2x_1 & -2x_2 & +3x_3 & = & 0 \\ 3x_1 & +x_2 & +4x_3 & = & -3 \\ -2x_1 & +3x_2 & -5x_3 & = & 3 \end{array} \right. \qquad (2)\ \left\{ \begin{array}{rrrcr} ax_1 & +bx_2 & +cx_3 & = & 1 \\ cx_1 & +ax_2 & +bx_3 & = & 1 \\ bx_1 & +cx_2 & +ax_3 & = & 1 \end{array} \right.$$

46. Let A be a square matrix with integral entries. Show that a necessary and sufficient condition for A to have an inverse with only integral entries is that $\det A = \pm 1$.

47. Show that the diagonal entries of a Hermitian matrix (a complex matrix satisfying that $A^* \equiv \overline{{}^tA} = A$.) is real numbers.

48. Let X, Y, Z, F be $n \times n$ matrices, and define $[X, Y]_F = XFY - YFX$. Then prove the following. (1) $[[X, Y]_F, Z]_F + [[Y, Z]_F, X]_F + [[Z, X]_F, Y]_F = O$

(2) If F is symmetric (i.e. ${}^tF = F$), and X and Y are skew-symmetric (i.e. ${}^tX = -X$, ${}^tY = -Y$), then $[X, Y]_F$ is skew-symmetric.

49.* Let A be an $l \times m$ matrix and B be an $m \times n$ matrix, then show that the rank of AB does not exceed the rank of A nor the rank of B.

50. Determine whether the following set V is regarded as a real vector space or a complex vector space. Answer with reasons.

(1) $V = \left\{ \begin{pmatrix} x \\ yi \\ 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ (2) $\left\{ \begin{pmatrix} xi \\ y \\ z+zi \\ w-wi \end{pmatrix} \mid x, y, z, w \in \mathbb{R} \right\}$

(3) $\left\{ \begin{pmatrix} x+yi \\ x-yi \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ (4) $\left\{ \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$

50.1 Show that several vectors containing $\mathsf{0}$ are linearly dependent.
50.2 Show that several vectors containing at least two identical vectors are linearly dependent.
50.3 Show that the following two conditions (i) and (ii) are equivalent
(i) The vectors $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ are linearly dependent.
(ii) One of the vectors $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ is expressed as a linear combination of the other vectors.
50.4 Show that if $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_n$ are linearly independent, then any vectors a_{i_1}, $\mathsf{a}_{i_2}, \ldots, \mathsf{a}_{i_s}$ selected from them are also linearly independent.
51. Which are bases of $V^3 (= \mathbb{R}^3)$? Answer with reasons.

(1) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ (2) $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

(3) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ (4) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

52. Which are subspaces of V^3? Answer with reasons.

(1) A plane $x + 2y - 3z = 0$ (2) A plane $x + 2y - 3z = -4$
(3) A plane $2x - y - z = 0$
(4) A line $x - 3 = y - 1 = z/2$
(5) A line $x/2 = -y/3 = -z$ (6) $\{\mathsf{0}\}$ (7) V^3

53. [52.] Denote by W_1, W_2 and W_3 the subspaces (1), (3) and (5) above, respectively.

(1) Find a basis of W_1 (2) Find a basis of W_2
(3) Find a basis of W_3

54. Given the following subspaces W, X, Y of $V^3(=\mathbb{R}^3)$ as follows, determine bases of the subspaces: (1) W, (2) X, (3) Y, (4) $W+X$, (5) $X+Y$, (6) $Y+W$, (7) $W\cap X$, (8) $X\cap Y$, (9) $Y\cap W$.

$$W=\left\{\begin{pmatrix} x\\ y\\ z\end{pmatrix}\in V^3 \mid \begin{matrix} x+y=0\\ x+y+z=0\end{matrix}\right\},\ X=\left\{\begin{pmatrix} x\\ y\\ z\end{pmatrix}\in V^3 \mid x+2y-z=0\right\},$$

$$Y=\left\{\begin{pmatrix} x\\ y\\ z\end{pmatrix}\in V^3 \mid \begin{matrix} 2x+y-z=0\\ x-y=0\end{matrix}\right\}.$$

(10) Which of (4), (5), (6) are direct sums?

55. Determine a basis of the following subspace W of $\mathbb{C}^4$.

$$(1)\ \left\{\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix} \mid x+iy=z+iw\right\}\quad (2)\ \left\{\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix} \mid \begin{matrix} x+iy=z+iw\\ x=2iy+3z+4iw\end{matrix}\right\}$$

56.* Given 3-dimensional vectors: $\mathtt{a}=\begin{pmatrix} 1\\ 3\\ 5\end{pmatrix}$, $\mathtt{b}=\begin{pmatrix} -1\\ 0\\ 1\end{pmatrix}$, $\mathtt{c}=\begin{pmatrix} 1\\ 1\\ 4\end{pmatrix}$, $\mathtt{d}=\begin{pmatrix} 2\\ 3\\ -1\end{pmatrix}$. Let V be the subspace spanned by $\mathtt{a}$ and $\mathtt{b}$, let W be the subspace spanned by $\mathtt{c}$ and $\mathtt{d}$. Determine a basis of $V\cap W$.

57. (1) Let $V=V^3$ and $W=\{\text{all } 2\times 2 \text{ real matrices}\}$, then V and W are regarded as real vector spaces. Define a mapping $T: V\longrightarrow W$ by $T\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} x & y\\ y & z\end{pmatrix}$. Prove that T is a linear mapping.

(2) Let $Y=\{\text{all } 2\times 2 \text{ real symmetric matrices}\}$. Prove that Y is a subspace of W.

(3) Define a mapping $T: V\longrightarrow Y$ by (1). Prove that this T is an isomorphism.

58. Prove that the composition ST of linear mappings T and S is also a linear mapping.

59. Prove that for a linear mapping $T: V\longrightarrow V'$, (1) the image and (2) the kernel of T are subspaces of V' and V, respectively.

59.1 Prove that a linear mapping $T: V\longrightarrow V'$ is determined by the images of all vectors of a basis of V.

60. For an isomorphism T between vector spaces V and V', the inverse of T is also an isomorphism between V' and V. (If a linear mapping is bijective, then it is called an isomorphism. You may assume that the inverse of a bijection is also a bijection.)

61. Find bases of (a) the image and (b) the kernel of the linear mappings determined by the following matrices.

(1) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 0 & 1 & 2 \\ -2 & -1 & -2 & 1 \\ 0 & -1 & -1 & 3 \end{pmatrix}$ (3) $\begin{pmatrix} 3 & -2 & -1 & 2 \\ -1 & 2 & 3 & -2 \\ -2 & -1 & 2 & 3 \\ 2 & 0 & 2 & 0 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 4 & 1 & 0 \\ 0 & 1 & -3 & -3 & 1 \end{pmatrix}$

62. Determine the matrix A of a linear transformation T of V^3 which mappings every point to the symmetric point with respect to the following plane S. (1) $x + 2y - 2z = 0$. (2) $y = x$.

63. Let W be 2-dimensional subspace of V^3 composed of all vectors ${}^t(x, y, z)$ such that $x + y + z = 0$. We have two bases of W: $\mathbb{E} = \langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \rangle$, $\mathbb{F} = \langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rangle$.

(1) Determine the matrix P of the base change $\mathbb{E} \to \mathbb{F}$.

(2) Define a linear transformation T of W by $T\mathbf{x} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{x}$.

(a) Determine the matrix A of T with respect to $\mathbb{E}$.

(b) Determine the matrix B of T with respect to $\mathbb{F}$.

(c) Find a relation satisfied by P, A and B.

64. Let V be the real vector space consisting of all polynomials in t with real coefficients of degree at most 3. Let W be the real vector space consisting of all polynomials in t with real coefficients of degree at most 2. Let $\mathbb{E} = \langle 1, t, t^2, t^3 \rangle$ be a basis of V, and $\mathbb{F} = \langle 1, t, t^2 \rangle$ be a basis of W.

(1) Define a mapping $T : V \longrightarrow W$ by $T(p(t)) = -p'(1 - t)$. Show that T is a linear mapping.

(2) Determine the matrix A of T with respect to $\mathbb{E}$ and $\mathbb{F}$.

(3) Define a linear transformation $\tilde{T}$ of V by $\tilde{T}(p(t)) = (t + b)p'(t + a)$ $(a, b$ are constants). Determine the matrix B of $\tilde{T}$ with respect to $\mathbb{E}$.

65. Letting $V = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{C} \right\}$,

then V is a complex vector space. For a 2×2 matrix $S = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, define a linear transformation T of V by $TX = SXS$.

(1) Show that T is a linear transformation of V.

(2) Let $\mathbb{E} = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ be a basis of V. Determine the matrix A of T with respect to $\mathbb{E}$.

(3) Using a basis $\mathbb{H}$ of the image of T_A and basis $\mathbb{G}$ of the kernel of T_A, determine a basis $\tilde{\mathbb{H}}$ of the image of T and a basis $\tilde{\mathbb{G}}$ of the kernel of T.

66.* For $m \times n$ matrices A and B, show that $r(A + B) \leq r(A) + r(B)$.