AN INTERIOR SURFACE GENERATION METHOD FOR ALL-HEXAHEDRAL MESHING

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ABSTRACT

This paper describes an all-hexahedral generation method focusing on how to create interior surfaces. It is well known that a solid homeomorphic to a ball with even number of bounding quadrilaterals can be partitioned into a compatible hexahedral mesh where each associated hexahedron corresponds to the intersection of three interior surfaces that are dual to the original hexahedral mesh. However, no such method for creating dual interior surfaces has been developed for generating all-hexahedral meshes of volumes covered with simply-connected quadrilaterals. We generate an interior surface as an orientable regular homotopy (or more definitively a sweep) by splitting a dual cycle into several pieces at self-intersecting points and joining the three connected pieces, if the self-intersecting point-types are identical, while we generate a non-orientable surface (containing Möbius bands) if the self-intersecting point-types are distinct. Stitching these simple interior surfaces together allows us to compose more complex interior surfaces. Thus, we propose a generalized method of generating a hexahedral mesh topology by directly creating the interior surface arrangement. We apply the present framework to Schneiders' open pyramid problem, and show an arrangement of interior surfaces that decompose Schneiders' pyramid into 146 hexahedra.

Keywords: all-hexahedral mesh generation, interior surface arrangement, dual cycle, Schneiders' pyramid

1. INTRODUCTION

Product development in every industrial field is in a severe competitive phase, and Finite Element (FE) analysis plays an important role in product development. It is known that, in many FE applications, hexahedral meshes give better and more effective results than tetrahedral ones. However, due to the difficulty in generating hexahedral meshes, industrial interests are shifting to hex-dominant meshes (Owen 1999, 2001), even though all-hexahedral meshes still give better results than hex-dominant ones in FE analysis.

Though hexahedral meshes are most commonly generated using a sweeping or blocking approach, it is desirable to generate an all-hexahedral mesh for a model of arbitrary topological type, rather than using one of these primitive methods. However, a generalized algorithm for creating such meshes does not currently exist. Techniques such as plastering (Blacker and Meyer 1993), whisker weaving (Tautges and Mitchell 1995; Tautges et al. 1996; Folwell and Mitchell 1998), and dual cycle elimination (Müller-Hannemann 1998) have been developed, in attempt to realize such an algorithm. Unfortunately, none of them are considered to be reliable in practical use, because they can handle only a limited class of solids, and have no guarantees on the quality of the resulting meshes.

In 1995, Mitchell proposed a *hexahedral mesh existence theorem* based on an arrangement of interior surfaces bounded by dual cycles on the surface of the (target) solid (Thurston 1993; Mitchell 1996; Eppstein 1996). This theorem states that any simply-connected three-dimensional domain, with even number of quadrilateral boundary faces, can be partitioned into a hexahedral mesh respecting the boundary.

Despite Mitchell's theorem, no practical method for generating such interior surfaces has been revealed yet, and, despite the proof, we (and many others) have encountered several deadlocks in creating the topology of hexahedral mesh that conforms to a given quadrilateral boundary mesh. In this paper we present an interior surface classification theory, and a practical method that could be utilized to create actual interior surfaces respecting the given boundary constraints, together with some discussion as to how these surfaces introduce a hexahedral mesh. As our later examples show, even if a topological solution exists, it may not always be an acceptable solution. Therefore, we will also discuss strategies to identify in-appropriate quadrilateral meshes when generating a hexahedral mesh for practical use.

In this paper we focus on the properties of interior surfaces, and leave the representation of their arrangements as a future problem. The remainder of this paper is organized as follows. In Section 2, we review interior surfaces and their arrangements, and deduce the associated requirements for topologically sound hexahedral mesh generation. In Section 3, the classification of self-intersecting point-types is discussed. In Section 4, we describe a method of generating simple interior surfaces that have at most one pair of self-intersecting points. Section 5 discusses a possible solution for composing more general interior surfaces as a collection of the simple ones. In Section 6, we demonstrate a dual space solution of Schneiders' open problem based on the methods described in the previous sections. In Section 7, we conclude this paper together with a brief discussion of some future developments for acceptable hex mesh generation.

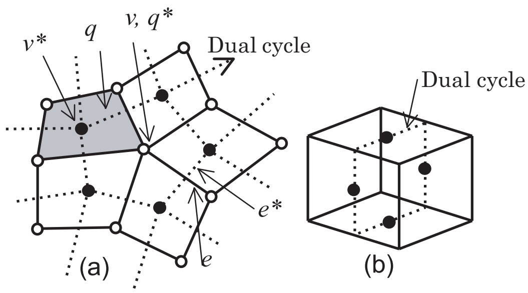
2. INTERIOR SURFACE ARRANGEMENT

2.1 Dual representation

In this paper we employ a technique based on the notion of an *interior surface arrangement* (Mitchell 1996), which has been used in the whisker weaving technique. For simplicity, we discuss hex meshing of solids homeomorphic to a ball, hereafter.

We denote a set of *vertices*, *edges*, *quads* (quadrilaterals) and *hexes* (hexahedra) as $V = \{v\}, E = \{e\}, Q = \{q\},$ and $H = \{h\}$, respectively.

For the *primal graph* $G_Q = G_Q(V, E)$ of a quad mesh M_Q, which is a planar graph because the solid is supposed to be homeomorphic to a ball, its *dual graph* $G_Q{}^* = G_Q{}^*(V^*, E^*)$ is composed as follows. A *dual vertex* $v^* \in V^*$ is placed inside each quad q, and a *dual edge* e^* connects the adjacent dual vertices if their corresponding quads share an edge in G_Q. A vertex $v \in G_Q$ is represented as a *dual face* q^* surrounded by dual edges in G_Q*. An edge $e \in G_Q$ corresponds to a dual edge e^* (Fig. 1a). For a quadrilateral mesh on a closed 2D manifold, the sequence of dual edges passing through opposing edges of each quad in G_Q is always closed as shown in Fig. 1b, since the quad mesh G_Q is assumed to be closed, the opposing edge of a quad is uniquely decided, and the number of its associated edges is finite. Thus the sequence of dual edge is called a *dual cycle* (Müller-Hannemann 1998) (Fig. 1b). A dual vertex is the intersection of two (local) dual cycles (Fig. 1a), which may include self-intersections of a single dual cycle.



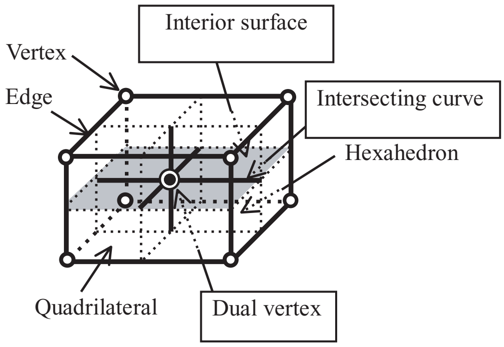


Fig. 1. (a) A Dual graph and (b) a dual cycle

Fig. 2 Dual representation of a hex mesh

Similarly, for the primal graph $G = G(V, E)$ of a hex mesh M_H, its dual graph $G^* = G^*(V^*, E^*)$ is constructed as follows. A dual vertex $v^* \in V^*$ is placed inside each hex h of G, and a dual edge *e** is placed between two adjacent dual vertices if their corresponding hexes share a quad in *G*. A hex *h*, a quad *q*, an edge *e* and a vertex *v* of a primal graph *G* correspond to a dual vertex *v**, a dual edge *e**, a dual face *q** surrounded by dual edges, and a dual polyhedron *h** enclosed by dual faces in dual graph *G**, respectively. A topological representation by dual vertices V^*, dual edges E^*, dual faces Q^*, and dual polyhedra H^* are called the *dual representation* $M_H{}^*$ of a hex mesh M_H.

A sheet-like layer of hexes corresponds to an *interior surface* (Eppstein 1996). A line, or column, of hexes corresponds to the intersection of two interior surfaces. A hexahedron is the dual to a vertex at the intersection of three interior surfaces in the dual representation of the hex mesh (Fig. 2).

2.2 Hexahedral mesh existence theorem

According to (Eppstein 1996), the hexahedral mesh existence theorem is described as follows:

Any simply connected three-dimensional domain with an even number of boundary quadrilateral faces can be partitioned into a hexahedral mesh respecting the boundary.

In an all-hexahedral mesh, each quad on the surface is always matched with another quad on the surface by tracking the connected dual edges inside the three-dimensional solid. Therefore, one of the necessary conditions for an all-hexahedral mesh is that the surface is covered with even number of quads. Thus, it can be shown that the number of self-intersections of dual cycles must also be even, since the intersection of two distinct dual cycles makes a pair of quads.

The proof steps of the hexahedral mesh existence theorem are as follows (Mitchell 1996, Thurston 1993):

1. The surface mesh of an object is mapped onto a sphere preserving its quadrilateral mesh connectivity,
2. The arrangement of dual cycles is extracted from the dual graph of the quadrilateral mesh on the spherical surface,
3. The arrangement of 2D manifolds is extended to an arrangement of 2D manifolds through the interior of the ball, (Smale 1958)
4. Additional 2D manifolds that are closed, and completely within the solid are inserted, if necessary, and
5. The arrangement of 2D manifolds is transformed to its dual to induce a hexahedral mesh.

The hexahedral mesh existence theorem, however, addresses only the existence and does not describe how to create interior surfaces using the dual cycles in Step 3. This paper proposes a method of creating an arrangement of interior surfaces from dual cycles, and provides a novel guideline for generating all hex meshes that takes advantage of not only the topological connectivity of the quad mesh elements but also their associated geometric information. We also give a solution for Schneiders' open problem as an example where the arrangement of interior surfaces is taken into account.

2.3 Requirements for interior surfaces

2.3.1 Boundaries

An interior surface must be either bounded by one or more dual cycles, or entirely closed inside the solid. In this paper we mainly discuss the former, while the latter appears to recover the topological convexity, which will be described later.

2.3.2 Regularity of homotopies

Here, we discuss the regularity of interior surfaces.

Let two functions $h_0(u), h_1(u), u \in I = [0,1]$ be continuous with respect to *u*. For a collection of continuous mappings $\{h_v(u)\}$, $H(u,v)=h_v(u)$ is called a homotopy if $H(u,v)=h_v(u)$ is also continuous with respect to $v \in J = [0,1]$ (Fig.3). We call a point $\boldsymbol{P}$ on a homotopy a *regular point,* if the image of its neighborhood, a 2D topological disc *D* becomes a disc *D*' (Fig.3). If any point on the homotopy *H* is a regular point, we call the homotopy *H* a regular homotopy, in this paper.

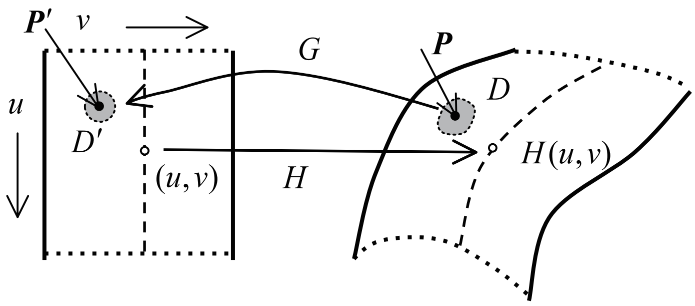


Fig. 3. Regular homotopy

Singular points, defined to be not regular points, can be illustrated by simple examples. Suppose we try to generate a homotopy *H*, as shown in Fig. 4a, between the two circles C_1 and C_2 lying on the two parallel planes π_1 and π_2 respectively, where the circles have opposite orientations. Then, the homotopy *H* contains two singular points, where no small smooth disc-like neighborhood can be formed (Refer to Whitney umbrella (Francis 1987 p.6)), or we can't find, on the homotopy, a small closed path without self-intersection around $\boldsymbol{P}_1$ or $\boldsymbol{P}_2$, but can find only letter-8-type one at most.

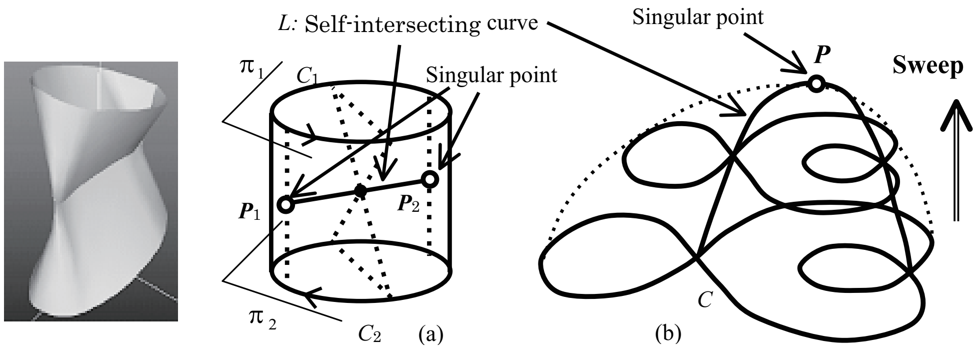


Fig. 4. Singular points

On the other hand, Fig. 4b depicts a homotopy created by sweeping the curve *C*. In this case, the singular point ***P*** exists on the top of the self-intersecting curve connecting the two self-intersecting points on the boundary curve at the bottom, where also no small smooth disc-like neighborhood can be formed, or we can find, on the homotopy, no small closed path without self-intersection around ***P***, since the neighborhood of ***P*** is a topological cone whose vertex and bottom are the point ***P*** and the curve homeomorphic to *C*, respectively

Regularity of homotopies is a necessary condition to create a hex element at any point on the interior surface, and we will select the types of interior surfaces based on the regularity of homotopy together with winding numbers mentioned later.

2.3.3 Self-intersecting point pair connectivity

As described in Section 2.2, two self-intersecting points on the boundary of an interior surface must always be paired, and connected by a self-intersecting curve. We will call this feature the *self-intersecting point-pair connectivity* in this paper.

The self-intersecting point-pair connectivity is a necessary condition for ensuring the regular homotopies between the boundary curves of the interior surfaces. Some self-intersecting curves of regular homotopies, however, may introduce unnecessarily complicated structures in the final hexahedral mesh. For example, as shown on the left of Fig. 5, the self-intersecting curve goes up and down several times, which results in an unexpected cross-sectional transition of the regular homotopy as shown on the right of Fig. 5, where the self-intersecting curve, which is represented by a bold line here, alternately changes its direction around the position indicated by the arrow. Although such an interior surface still preserves the topological condition for the hex mesh, it will undoubtedly reduce the quality of the hex mesh, especially when the radius of curvature of the self-intersecting curve is small. This undesirable case of interior surface arrangements should be avoided when generating geometically reasonable hex meshes.

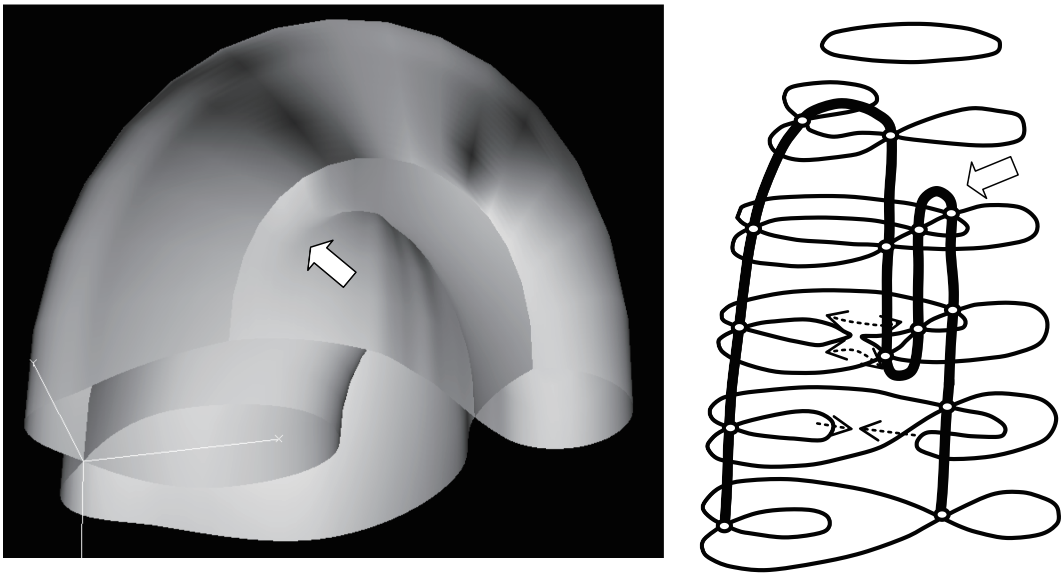


Fig. 5. A regular homotopy between the top and bottom curves where a self-intersecting curve introduces complicated geometrical structures

2.4 Requirement for interior surface arrangement

2.4.1 Topological Convexity

It is well-known that hex elements used in FE analysis must be convex. Fig. 6a and 6b show representative examples where two quads share two edges in hex meshes, which results in non-convex elements in a topological sense. A hex mesh is *topologically convex*, unless any of its two quads share two or more edges in the primal mesh. Ensuring the topological convexity of a hex mesh M_H is equivalent to proving that the dual graph $G_F*(F*, E*)$ composed of the dual face set $F*$ and the dual edge set $E*$of M_H is simple, where the simplicity of a graph means that the graph has no self-loops or multi-edges. Fig. 6 also includes the dual representations of the non-convex hex meshes, where the two dual faces f_1* and f_2* are incident to the two common dual edges e_1* and e_2*.

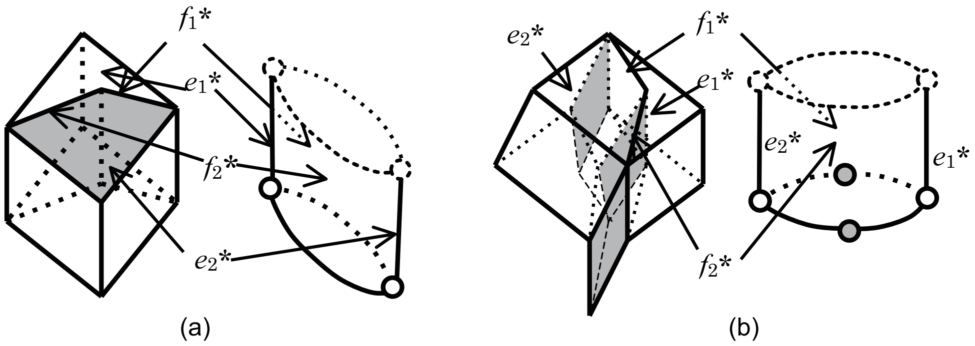


Fig. 6. Topologically non-convex hexes (Note that the hexes corresponding to the dotted dual vertices are omitted.)

Note that the simplicity of a dual graph $G_V*(V*,E*)$ consisting of the dual vertex set $V*$ and the dual edge set $E*$ of M_H does not always ensure the simplicity of the dual graph $G_F*(F*,E*)$. For example, though the dual graph $G_V*(V*,E*)$ of Fig. 6b is simple, its dual graph $G_F*(F*,E*)$ is not simple as shown with gray circles.

A convexity recovery operation by a hex layer insertion is depicted in Fig. 7. It should also be noted that an all-hexahedral mesh bounded by a convex quad mesh is not always topologically convex.

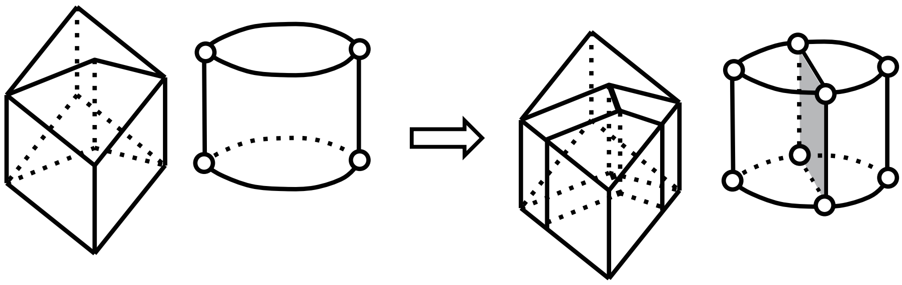


Fig.7. Operation for recovering topological convexity

2.4.2 Hexahedral element existence and connectivity

There are other necessary conditions to form a valid topology.

There must be at least one dual vertex on every curve intersecting with two interior surfaces for constructing a valid all-hexahedral mesh. In addition, any two hex elements in a hex mesh must be connected through (a single quad or) a sequence of quads. In other words a dual graph of a hex mesh must be connected.

2.4.3 Tangent Problem

In the above discussion it is presumed that regular homotopies have no tangent region (see Fig. 8), which is defined to be a point/region with the tangent shared by two (local) regular homotopies.

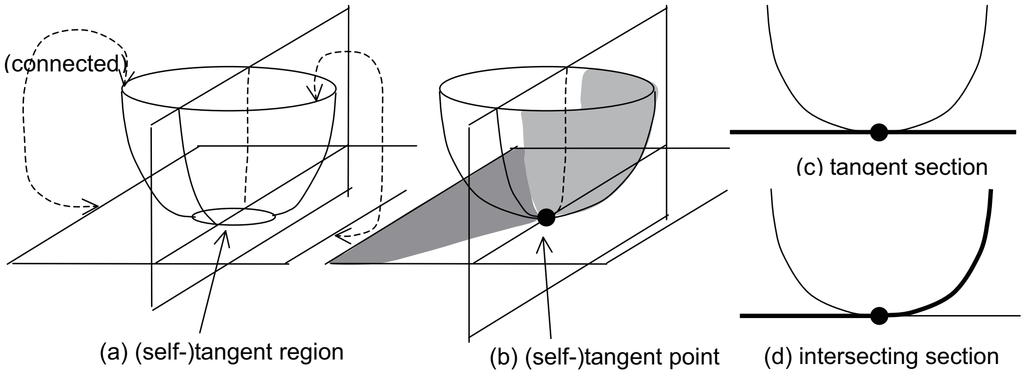


Fig. 8. Homotopies with (self-)tangent point/region

It is clear that such a (self-)tangent area (Fig. 8a) can be reduced to be a (self-) tangent point as shown in Fig. 8b. Here, we switch the connectivity of four local interior surfaces that meet at the (self-)tangent point (Fig. 8c) so that the two resultant surfaces actually intersect with each other (Fig. 8d), while the associated topological structure of the polyhedrons is unchanged. Thus, we can conclude that a (self-)tangent case of regular homotopies can be transformed to a simple (self-) intersection case that we have discussed earlier.

Hereafter we will discuss only (self-)intersecting regular homotopies without (self-)tangent points.

3. SELF-INTERSECTING POINT TYPE

Any two curves with even number of self-intersecting points are topologically equivalent on a sphere (Fig. 9). However, the two surfaces bounded by the curves cannot always be deformed into each other. For example, as shown in Fig. 9, the surface bounded by the first curve is orientable, while one by the last curve is non-orientable. These two surfaces are not deformable into each other under the constraint of regular homotopies. In this section the identification of self-intersecting point-types are discussed in order to select the appropriate type of interior surface to be generated. Not only orientable surfaces but also non-orientable[1] surfaces such as Möbius bands and their connected sum can serve as interior surfaces (Schwartz and Ziegler 2004), and, in practice, the self-intersecting point-types are closely related to the orientability of its associated interior surfaces.

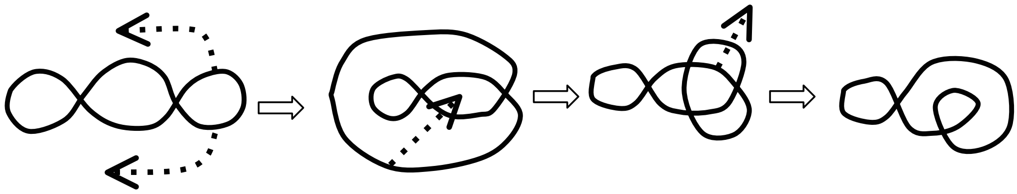


Fig. 9. Topological deformation (transformation) between two curves with even number of self-intersecting points on a sphere

A convex polyhedron is can be mapped onto a plane by removing a face called a *window* and rolling out its boundary as shown in Fig. 10, where the projected planar graph *G* (solid line) and its dual *G** (dotted line) are presented. As shown in Fig. 11, the *winding number* $w(\boldsymbol{P},C)$ of the point $\boldsymbol{P}$ for the closed oriented curve C on a plane is defined as the (signed) number of times a point $\boldsymbol{Q}$ on C passes counterclockwise around the point $\boldsymbol{P}$ (Fig. 10). A dual cycle mapped on a plane divides the plane into several regions. There are 4 regions incident to a self-intersecting point partitioned by the curve, and the winding number of any point within each region is identical.

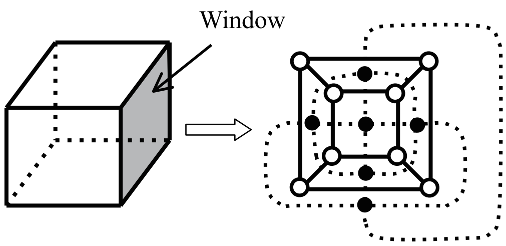


Fig. 10. A cube and its planar version together with the corresponding dual graph

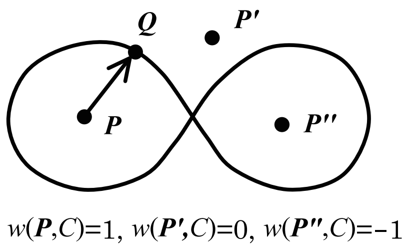


Fig. 11. Winding number of a point for a curve

Figs. 12 and 13 show dual cycles with a pair of self-intersecting points together with the winding numbers of the partitioned regions around the self-intersecting points. The winding numbers of the regions around a self-intersecting point can be

[1] While orientability is basically defined for only a closed surface, we can inherit and consequently define the 'orientability' of an open surface by gluing topological disks along its boundary circles to refer to the orientability of its corresponding closed surface.

represented as (*i*, *i*+1, *i*+2, *i*+1), where *i* is called the *minimum order*, here. If the minimum winding numbers of the paired self-intersecting points are equal, then the *self-intersecting point-type* of the (1-simple) dual cycle is said to be identical (Fig. 12), otherwise they are distinct (Fig. 13).

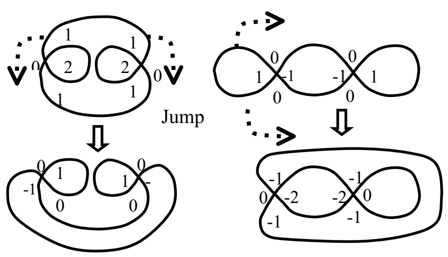


Fig.12. Identical self-intersecting point-type

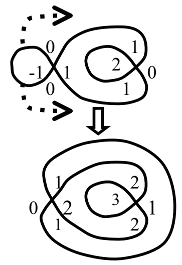


Fig.13. Distinct self-intersecting point-type

There are three important points to note. A portion of a dual cycle can *jump* over the paired self-intersecting points without crossing over it (see Figs. 11 and 12). This does not change the self-intersecting point-type. Secondly, a dual edge jumps if the window crosses over the edge composing the dual cycle. This also does not change the self-intersecting point-type. Thirdly, the self-intersecting point-type is changed if an edge *crosses over* one of the self-intersecting points (see Fig. 14).

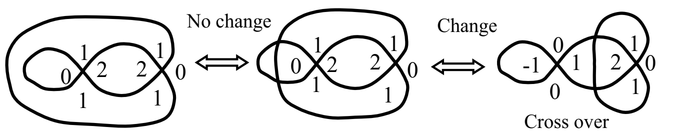


Fig.14. Self-intersecting point-type changes when an edge crosses over a self-intersecting point

To generate valid interior surfaces, the two boundary dual cycles in an identical interior surface should be connected. This is due to the fact that although the winding number of a region must be identical, we may potentially obtain erroneous winding numbers for non-connected dual cycles, if the selection of the window is not appropriate. For example, as shown in Fig. 15, as mentioned above a change of window causes jump, which results in incompatible winding numbers.

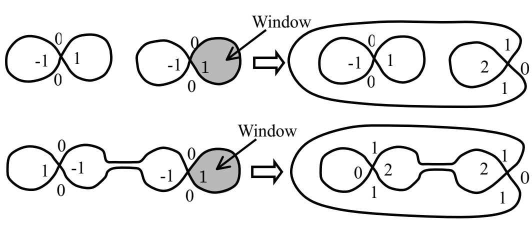


Fig. 15. A change of a window causes erroneous winding numbers for not connected dual cycles

4. 1-SIMPLE INTERIOR SURFACES

In this section we will describe a technique to create "1-simple" interior surfaces from the dual cycles containing, at most, a single pair of self-intersecting points. In the next section, we will describe a technique for generating interior surfaces of two or more pairs of self-intersecting points by extending the notion of the "1-simple" interior surface to be described in this section.

4.1 1-simple interior surfaces bounded by a single dual cycle

In this subsection, we describe a technique to create a *1-simple interior surface* bounded by a single dual cycle. We define an interior surface as “1-simple” if its bounding dual cycle(s) has (have) at most a single pair of self-intersecting points in total (Fig. 16).

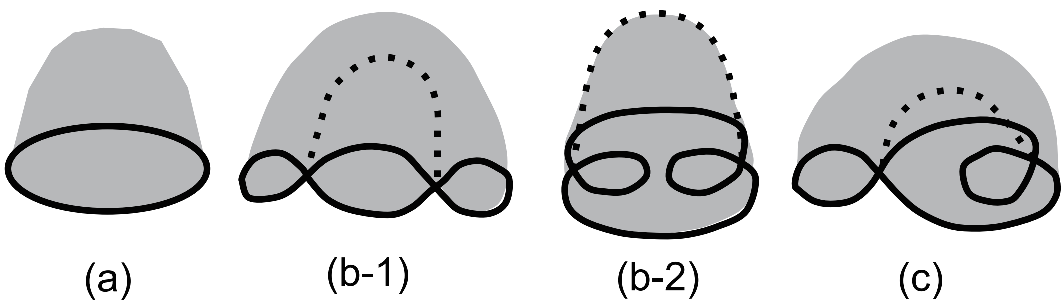


Fig. 16. 1-simple interior surface

Based on Smale's theorem (Smale 1958), Mitchell pointed out (Mitchell 1996) that there exists a regular homotopy between a closed curve with even number of self-intersecting points and a closed curve without self-intersections. According to this criterion, the self-intersecting point-type is classified into the following three cases (Fig. 16):

(a) No self-intersection,
(b) Identical self-intersecting point type, or
(c) Different self-intersecting point type.

From an implementation point of view, however, even if the existence of an interior surface might be proven, it is not always appropriate to follow the method described in the proof for creating the interior surface along the given dual cycles. For example, as shown in Fig.18, we can indeed generate interior surfaces as a homotopy between the given cycle and a circle. However, this method is not familiar in CAD community. Therefore, we have to develop a practical method for creating 1-simple interior surfaces bounded by a single dual cycle.

However, before we consider the methods of generating the interior surfaces, let us first point out some constraints that will be helpful in guiding the practical creation of interior surfaces that fully respect the quality of a resultant hex mesh:

1. The interior surface should be connected to a dual cycle at nearly right angles with respect to the boundary surface of the given solid.
2. The interior surface should be smooth.
3. The interior surface must be completely embedded within the solid.
4. The self-intersecting curve connecting a pair of self-intersecting points on the dual cycle should be smooth. It is also desirable that this curve will not have many turning points to avoid multiple changes in their advancing direction (e.g. The case in Fig. 5 should be avoided.).
5. If the interior surface self-intersects, the angle of self-intersection should be nearly a right angle along the self-intersecting curve.
6. No self-intersections within the interior surface should be allowed, with the exception of the possible self-intersecting curve connecting a pair of self-intersecting points on the dual cycle.
7. The absolute value of the curvature of interior surfaces should be as small as possible.

Following the classification of the self-intersecting point-types for the 1-simple interior surface (i.e. a, b, or c of Fig. 16), we will consider how to incorporate the above-mentioned requirements for generating high-quality interior surfaces.

(a) Case of a dual cycle without self-intersecting points

If a dual cycle bounding a solid contains no self-intersecting points, the interior surface is represented as a swept type surface. The successive dual cycle elimination method tries to eliminate the surface layers from the outside one by one (Müller-Hannemann 1998). However, the elimination of the interior surface is limited to the surface of the current solid through the elimination process.

More generally, we can use an "*advancing closed curve method*" that continues advancing the closed curve roughly toward the normal direction of the surface, adjusting its size, and finally filling the disk when the closed curve becomes small enough (Fig. 17).

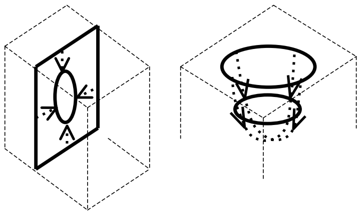


Fig.17. Advancing closed curve

(b) Case of a dual cycle with self-intersecting points of identical

According to Mitchell's proof of the hexahedral mesh existence problem, dual cycles with the identical self-intersecting point-type bound interior surfaces as illustrated in Fig. 18. As seen in this figure, it is difficult to control the shapes of these interior surfaces without taking into account of the self-intersecting points. In this paper, we propose a systematic algorithm for constructing the interior surfaces, to allow better control of the corresponding shape.

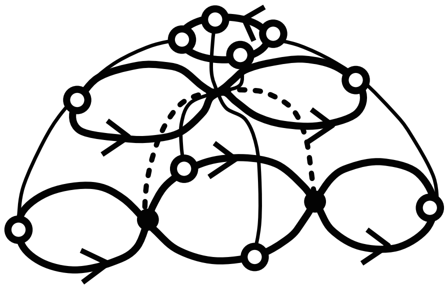


Fig. 18. Interior surface bounded by a dual cycle with identical self-intersecting point-type

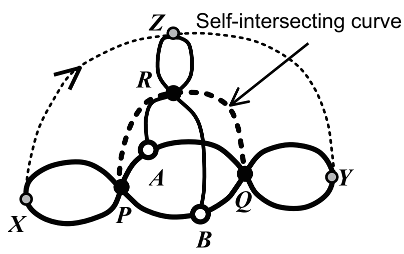


Fig. 19. Orientable interior surface generated by sweeping a segment on the dual cycle

Suppose the self-intersecting points ***P*** and ***Q***, have an identical self-intersecting point-type as shown in Fig.18. Here, the self-intersection points have symmetric arrangement of winding numbers in their incident partitioned regions. Let us consider a curve (represented as a thin solid line in Fig. 19) passing through the points ***A***, ***R***, ***Z***, ***R***, and ***B*** on the two *paths* connecting the self-intersecting points ***P*** and ***Q***. An interior surface is then generated by splitting the dual cycle into two segments ***APXPB*** to ***AQYQB*** and then sweep a sectional curve from one half to the other, by way of the above mentioned curve ***ARZRB***, which implies that the resultant swept surface is regular and orientable, because this is accomplished by sweeping a simple curve without any deformations. This special type of surface is a *rotational sweep* of the sectional curve ***ARZRB*** whose rotational axis passes through the points ***A*** and ***B*** (*pivots*) on the two paths, which will be utilized to solve Schneiders' problem later. Although we use the fixed pivots here for simplicity, we can also allow the pivots to move and obtain more general type of sweep.

(c) Case of a dual cycle with self-intersecting points of distinct types

Finally, we consider the most difficult case, i.e. an interior surface bounded by a dual cycle with intersecting points of distinct types (Fig. 16c). Actually, in this case, we introduce a Möbius band and match its boundary circle with the dual cycle to generate the interior surface as shown in Fig. 20. Note that the boundary of a Möbius band becomes a single closed curve as shown in (Francis 1987 p.118), because it can be formed by cutting a Klein bottle with a plane.

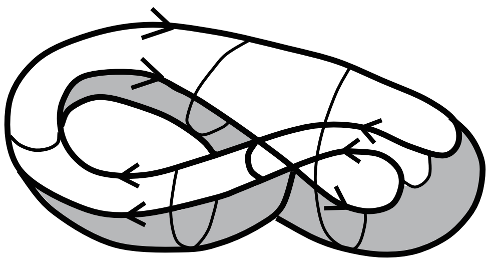


Fig. 20. A Möbius band obtained by splitting a Klein bottle

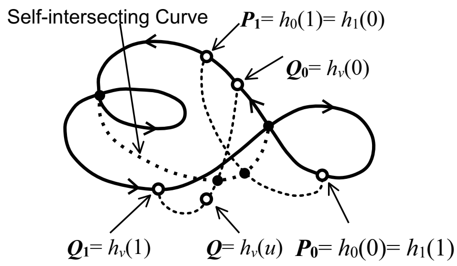


Fig. 21. Generation of a Möbius band

Thus, according to our classification, the resultant interior surface is regular and non-orientable in this case, because the two sides of this strip are no longer distinguishable. It is known that a non-orientable surface containing Möbius bands cannot be embedded into 3D space without self-intersections laying its boundary on a plane as illustrated in (Francis 1987 p.118),.

Fig. 21 demonstrates how to form a Möbius band from a dual cycle with a distinct self-intersecting point-type, where the associated regular homotopy is defined such that the points $\boldsymbol{P_0}$ and $\boldsymbol{P_1}$ are $h_0(0)(=h_1(1))$ and $h_1(0)(=h_0(1))$, respectively.

For this case, it is quite difficult to fully control the shape of a Möbius band only with the geometric positions of the ends $h_0(0)$ and $h_1(1)$ and their corresponding tangent vectors. Recall that, in the case of an interior surface of a Möbius band shown in Fig. 5, the self-intersecting curve changes its vertical orientation 3 times. However, in this case, we can reduce the number of orientation changes to one according to the sectional transition of the dual cycle shown in Fig. 22 by introducing a saddle point (the second from the left of Fig.22) with respect to the vertical axis. Even though this provides a topological solution for the case in Fig.16c, generating non-orientable interior surfaces containing Möbius bands should be avoided as much as possible.

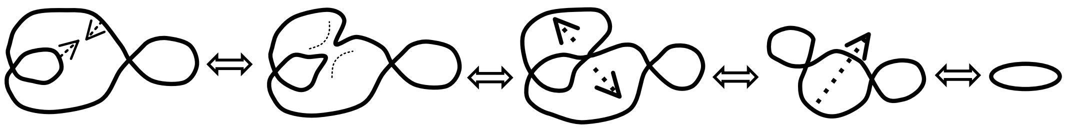


Fig. 22. Sectional transition diagram of a Möbius band to suppress the directional alternation

4.2 1-simple interior surfaces bounded by two dual cycles

We will now discuss a method of generating a 1-simple interior surface bounded by two dual cycles, where the number of self-intersections totals 0 or 2. If two dual cycles bound an interior surface, they should be connected prior to determining the self-intersecting point-type (Fig. 23), as described toward the end of Section 3. Once the two dual cycles are converted to a single closed curve, a regular homotopy can be obtained by the method described in Section 4.1. For the combinations of self-intersecting point-types together with the winding numbers, we can refer to the cases in Figs. 12 and 13 in Section 3

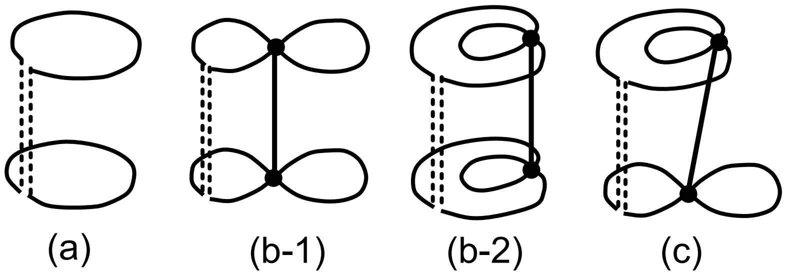


Fig. 23. Connection of two dual cycles

5. CONSTRUCTIVE GENERATION OF INTERIOR SURFACES

We propose a technique for constructively creating a more general interior surface where its corresponding dual-cycles have two or more pairs of self-intersecting points. The actual construction of interior surfaces will be accomplished by successively joining primitive surfaces called *basic interior surfaces* to form the more complex interior surfaces.

5.1 Decomposition into basic interior surfaces

In order to create an interior surface by connecting basic interior surfaces, it is necessary to identify the self-intersecting point-types. In this subsection, we define a basic interior surface and discuss the problem of determining the combinations of self-intersecting point-types.

5.1.1 Circuit, triple-circuit, and basic interior surface

We define a *circuit* to be a partial dual cycle if it contains no self-intersections except at its endpoints (Fig. 24). Here, we call the endpoints of a circuit a *base point*. Note that the intersections between different circuits are allowed as depicted in Fig.24. This definition allows us to iteratively identify new circuits on the dual cycle and remove them to define next circuits (Fig. 25).

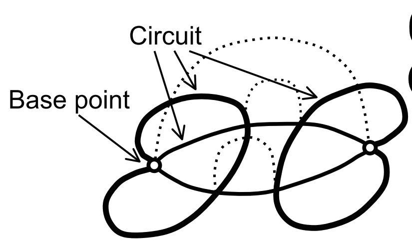


Fig. 24. A circuit and its base points

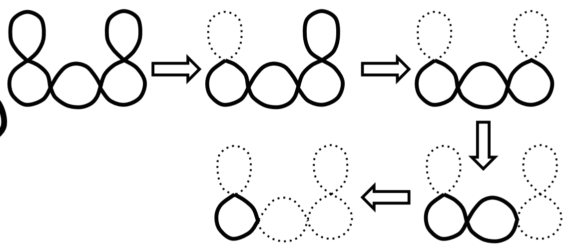


Fig. 25. Recursive identification of circuits

Recall that the 1-simple interior surfaces described in the previous section have, at most, two self-intersecting points. Therefore, we have to extend our method to handle dual cycles having more than two intersecting points. Let's call the tuple of three connected circuits a *triple-circuit* (Fig. 26). A triple-circuit may have more than two self-intersecting points of the dual cycle (e.g. 6 self-intersecting points in Fig. 26), while dual cycles that bound a 1-simple interior surface has, by definition, at most two.

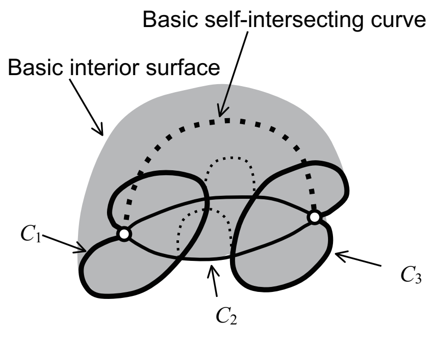


Fig. 26. Triple-circuit

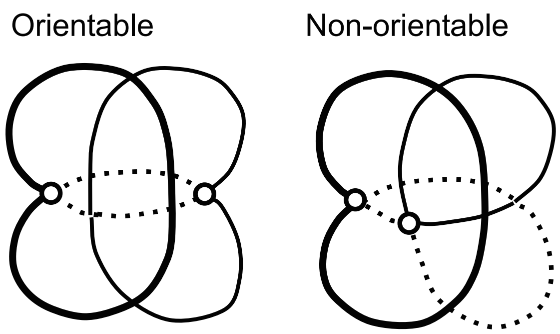


Fig. 27. Selection of a triple-circuit is not unique.

We call a 1-simple interior surface as a *basic interior surface* (Fig. 26) if it is created by a triple-circuit, and we also call the corresponding self-intersecting curve connecting the two base points as a *basic self-intersecting curve* (Fig. 26).

A basic interior surface bounding a dual cycle may be either “orientable” or “non-orientable”, because a selection of triple-circuits is not always determined uniquely from a given set of dual cycles. As shown in Fig. 27, an interior surface might be orientable (on the left) or non-orientable (on the right) even from the same dual cycle, depending on the selection of base points and their associated self-intersecting point-types. If the self-intersecting point-types of the selected base points are identical, the corresponding interior surface will be orientable, otherwise they will be non-orientable.

5.1.2 General Interior Surface Creation

Any interior surface can be created by iteratively creating basic interior surfaces (Fig. 28). We can demonstrate this by selecting a circuit shared by two triple-circuits (***ABCD*** in Fig.28), split the circuit by a joining edge (***XY***) that bridges two points on the circuit, create two basic interior surfaces respecting the updated triple-circuits, and join them. Note that any curve connecting the two base points through the interior surface is not permitted to intersect with other basic self-intersecting curves as shown in Fig. 29, because such intersecting points incur an illegal case where four local interior surfaces meet together, and thus prevent us from placing a valid hex element.

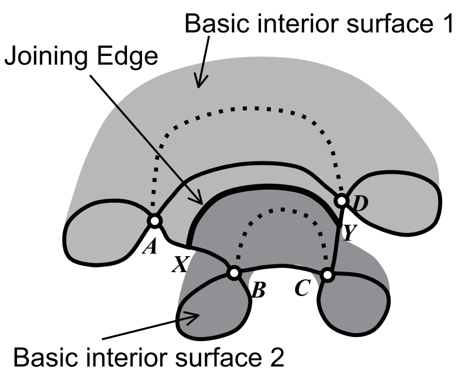


Fig. 28. Basic interior surface creation

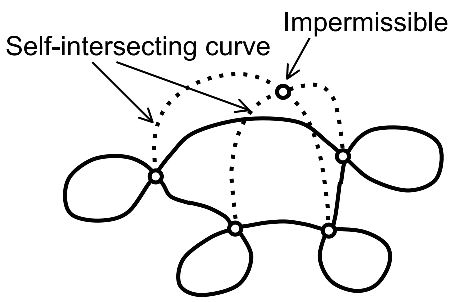


Fig. 29. An invalid case where self-intersecting curves intersect.

5.2 Secondary self-intersecting curves

Just after basic interior surfaces are created, only basic self-intersecting curves have been generated explicitly, though other self-intersecting curves may still exist intrinsically, for example the two thin dotted curves in Fig. 26. To generate valid topological data for the hex mesh, not only the information of the basic self-intersecting curves but also the *secondary self-intersecting curves* must be added. Secondary self-intersection may occur not only between distinct basic interior surfaces but also in a basic interior surface itself as depicted in Fig. 30, and will also appear in solving Schneiders' open problem. The process of interior surface creation is completed using this secondary self-intersecting curve calculation, which the process of creating the interior surface arrangement follows.

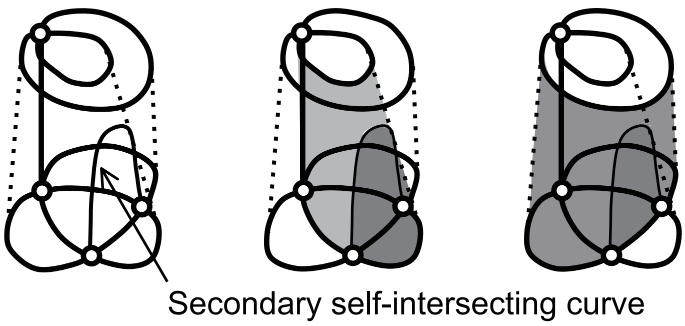
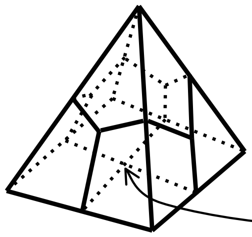


Fig. 30. Intersection in a basic interior surface

6. SOLUTION OF SCHNEIDERS' OPEN PROBLEM

Schneiders (Schneiders www) presents a problem regarding whether, or not, there exists a hexahedral mesh whose boundary exactly matches a pyramid with a prescribed surface mesh as shown in Fig. 31 (hereafter called "*Schneiders' pyramid*"). Though several solutions have been published for Schneiders' pyramid (Yamakawa and Shimada 2001), we will attempt to solve this problem in order to demonstrate that our "interior surface direct arrangement technique" by creating the interior surfaces directly in a dual space is generally applicable to the all-hexahedral mesh generation problem with generality.



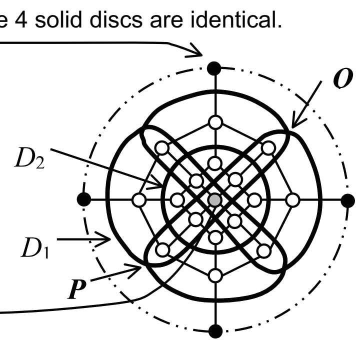
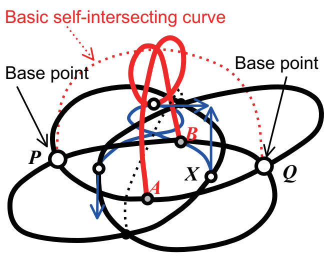


Fig. 31. Schneiders' pyramid **Fig. 32.** Triple-circuit for Schneiders' pyramid

The two dual cycles of Schneiders' pyramid, D_1 and D_2, are shown in Fig. 32. We select the points ***P*** and ***Q*** as the base points of the triple-circuit for the dual cycle D_1. (Note that this is the same as the dual cycle depicted in Fig. 27.) The base points are selected such that the self-intersecting point-types are identical so that we can obtain an orientable surface. Then, the interior surface can be created as a rotational sweep.



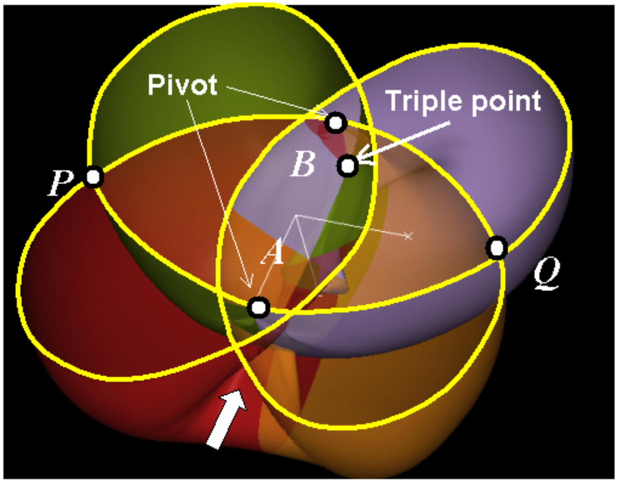
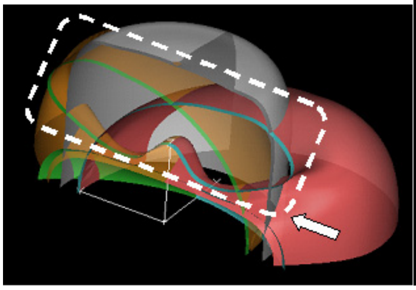


Fig. 33. Rotational sweep **Fig. 34.** Interior surface for Schneiders' pyramid

Let the sweep-section rotate from the base point ***P*** to the base point ***Q***, and let the points ***A*** and ***B*** be the pivots in Fig. 33. To obtain a regular homotopy, the sweep-vector is oriented upward at the starting point (left half), horizontally at the intermediate point (center; red), and downward at the end point (right half). In Fig. 33 the traveling directions are shown with blue arrows. Therefore, the trajectory of the point ***X*** on the sweep-section becomes a curve with a loop in blue. Fig. 34 shows the interior surface created with this curve.

This interior surface has four features. The first is that it is regular. The second is that that it is orientable. The third is that it has two triple points. (However, it is not a Boy's surface (Francis 1987 p.90) as often referred (Bern and Eppstein 2002) , because this surface is orientable.) The fourth feature is "*through hole*" whose section is shown in Fig.34 with an arrow or in Fig. 35 as a sectional view, which makes self-loops and increases the number of hexes needed to fill the volume considerably. Note that the through hole appears due to the intersection of rotational sweep section.



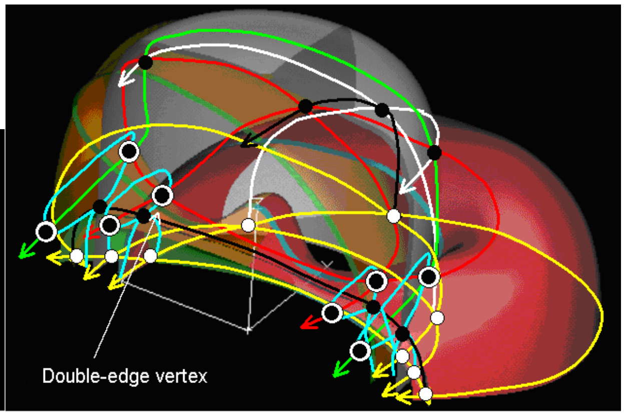


Fig. 35. Section of the interior surfaces **Fig. 36.** Intersection between the interior surfaces

Fig.35 shows a section of the interior surfaces for Schneiders' problem. Another interior surface bounded by the dual cycle D_2 is shown in Fig. 35 as a grey cylindrical surface. Each of the triple-intersection points represents a dual vertex, and the intersecting curves are the dual edges in the dual graph of the resulting mesh. Because of the four self-loops, several of the resulting hexes require their convexity to be recovered.

In order to recover the topological convexity, we insert a cylindrical (closed) surface (depicted with dotted line as a closed interior surface in Fig. 35). The resulting intersecting curves between the interior surfaces are added to the dual graph, along with the other symmetric half of the original surface, which produces 20 dual vertices (hexes) as can be seen in Fig. 36. The dual graph was represented with a prototype 3D cell-complex model, and converted to a primal graph, whose NASTRAN format data is shown in Tab.1 in the appendix together with the node numbers on the surface and the illustration Fig. A1.

Topologically, there are eight dual vertices having double edges shown as double discs in Fig. 36, where dual vertices on the boundary are shown as white discs, and inside ones are shown as black ones. We can recover the convexity for these dual vertices by adding another cylindrical surface, resulting in an additional 18 hexes. Furthermore, still there exist 9 convexity-taking edges, and this recovery requires 9 additional cylinders, which consequently create additional 90 hexes. For our interior surface arrangement it is confirmed that 146 dual vertices (hexes) complete the hexahedral mesh topology. However, this does not guarantee geometric convexity.

As shown above, a hex mesh can be generated from a closed quad mesh with complicated dual cycles using an "interior surface direct arrangement technique".

Schneiders' pyramid demonstrates to us that there exist quad meshes where there is an obvious interplay between interior surfaces that results in a poorer quality mesh. The process of recovering the topological convexity may intrinsically require the addition of many hex elements, which inevitably reduces the industrial value of the solution.

7. CONCLUSION

7.1 Our contribution

Indeed, the hexahedral mesh existence theorem, that "Any simply connected three-dimensional domain with an even number of quadrilateral bounding faces can be partitioned into a hexahedral mesh respecting the boundary (using the arrangement of interior surfaces extended from dual cycles into solids)" is well known. However, a practical algorithm for generating the meshes indicated by the proof has not yet been realized. Our contribution to all-hexahedral meshing is to provide with a solution for this problem, and can be summarized as follows:

We have introduced the notion of self-intersecting point-types for a dual cycle, and deduced that an interior surface becomes orientable and thus we generate it by a sweep operation if we pair self-intersecting points of identical type, otherwise the surface will be non-orientable because it inevitably contain Möbius bands. We have shown, for any dual cycle, an algorithm to form an interior surface constructively uniting basic interior surfaces. Thus, we have proposed a method of generating a hex mesh topology from any even numbered closed quad mesh by creating the interior surface arrangement.

We have created an interior surface for Schneiders' pyramid, and showed that it can be decomposed into 20 hexes, some of which are topologically non-convex ones, directly from the arrangement of interior surfaces, or into 146 hex elements when the topological convexity is recovered.

7.2 Future Problems

7.2.1 Unstructured all-hexahedral meshing

The following three questions concerning unstructured all-hexahedral mesh generation are important, and an algorithm that positively answers all three will provide an industrially viable solution.

1. Is the topological solution feasible?
2. Can the convexity of all hexes be recovered only from the topological arrangement of interior surfaces with a reasonable number of hexes?
3. Does the set of interior surfaces together with geometrically sufficient quality guarantee that the resultant hexahedral mesh also has sufficient quality?

The solution presented in this paper demonstrates an answer to the above-mentioned first question only for the example of Schneiders' pyramid. Further research and development will be needed for general applications.

The second question of convexity recovery for the hexes often leads to large increases in the number of hex elements, and is especially apparent when dealing with Möbius bands resulting from distinct self-intersecting point-types, as well as by “through holes” owing to intersection of the two end circuits of a tri-circuit with multiply self-intersecting interior surfaces. These problems, however, have been avoided in the past by enforcing utilizing hex-dominant mesh generation techniques and/or restricted unstructured hex mesh generation techniques that generate hex meshes in a much smaller geometric and topologic domains. Further research and development will be needed for this problem

The third question of how the geometric quality of the interior surface results in hex mesh quality deterioration has had little study, especially on the relationship between surface quad meshes and the interior surface generation requirements in Section 4.1. This is also an area of future research.

7.2.2 Implementation issues

The interior surfaces shown in this paper were generated without referring to the given volumetric space, and the induced topology was then mapped back into the volume. Because of the difficulty of generating interior surfaces in pre-defined volumetric boundaries (which would be required for sampling quality metrics of the resulting mesh that such interior surfaces might generate), a technique for directly arranging interior surfaces will pose some difficult implementation issues. For example, surface creation with boundary constraints, geometrical surface evaluation, topological representations of the induced interior surface arrangement, etc. should be investigated. Solving these problems is also an interesting theme for future research.

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APPENDIX

Table 1. The 20-Hexahedral mesh (convexity lost) by NASTRAN CHEXA Format

Hex	N 1	N 2	N 3	N 4	N 5	N 6	N 7	N 8
17	1	2	3	4	5	6	7	8
18	9	10	11	12	1	2	3	4
19	13	14	15	16	9	10	11	12
20	17	18	19	8	13	14	15	16
21	20	3	11	15	8	7	21	19
22	22	17	8	5	9	13	23	1
23	7	8	20	3	24	25	26	27
25	28	4	3	27	25	8	7	24
26	27	1	4	28	24	5	8	25
27	29	12	11	30	28	4	3	27
28	30	9	12	29	27	1	4	28
29	31	16	15	32	29	12	11	30
30	32	13	16	31	30	9	12	29
31	25	8	19	33	31	16	15	32
32	33	17	8	25	32	13	16	31
33	3	20	15	11	27	26	32	30
34	1	9	13	23	27	30	32	26
35	19	15	20	8	33	32	26	25
36	17	8	23	13	33	25	26	32

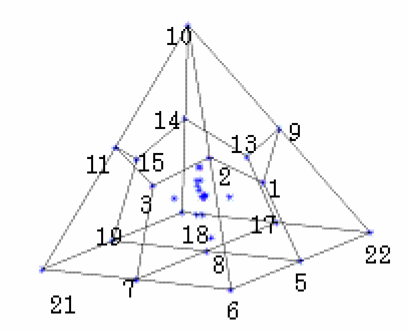

Fig. A1. Node numbering on the surface